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Complements of Higher Mathematics

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Preface

It is our belief that this book will help many students and researchers with mastering applied problems from different fields. The material presented here stems from courses and seminars on special topics in mathematics given by the authors for many years at universities, especially at technical, engineering, and economics faculties. We limit our considerations in this chapter to the basic notions and results of each domain of mathematics considered in this volume due to the diversity of the included chapters. Chapter 1 is devoted to complex functions. Here, emphasis is placed on the theory of holomorphic functions, which facilitate understanding the role played by the theory of complex variables in mathematical physics, especially in the modeling of plane problems. Furthermore, we consider the remarkable importance that the theories of special functions, operational calculus, and variational calculus have. A great part of the book is dedicated to second-order partial differential equations, since they are widely used to model phenomena in physics and engineering. In the last chapter, we discuss the basic elements of one of the most modern areas of mathematics, namely the theory of optimal control. To achieve a relative independence of the book, each chapter introduces the necessary mathematical background, i.e., topics from mathematical analysis, topology, functional analysis, and so on, which are used in the other chapters. For this reason, the larger part of this book is accessible to students of technical, engineering, and economics faculties, and researchers working in these fields. Some applications are included to illustrate the theory discussed. The methods used in the book permit the analysis of both theoretical and practical cases, thus offering results of interest to the students of technical, engineering, and economics faculties and, also, for young researchers, interested of higher mathematics.

Our intention was to help the reader to proceed more easily to the study of special topics in mathematics, which is usually studied in the second year of all technical faculties. A number of supplementary topics included in this book have been chosen particularly in consideration of their use in specialized courses.

For the study of this book, it is sufficient for the reader to be familiar with a classical course on mathematical analysis and a standard course on differential geometry and algebra, which usually are included in the first year of most programs.

The authors are aware that there are many more results and even more recent data regarding the domains which are not presented herein.

Only their simple enumeration, even in a simplified form to become more accessible, would have considerable enlarged the present book.

Excluding these allowed the authors to present thorough mathematical proofs of the results presented in the book.

The authors would be grateful for readers' comments on the content and the design of the textbook. We would also be pleased to receive any other suggestions the readers may wish to make.

We express our profound gratitude to Prof. C. Marinescu of the Department of Mathematics, Transilvania University of Brasov, for his kindness in reading the manuscript and making pertinent observations, which were taken into consideration.

We are grateful also for suggesting to write this book with the purpose of supplying the bibliographical material for students interested in higher mathematics.

Brasov, Romania — Marin Marin
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Contents

Chapter 1
Complex Functions

This chapter contains the basic results on complex functions of real as well as of complex variables. For more details and more results, the readers are referred to the books in the bibliography section.

1.1 Complex Functions of Real Variable

Let us consider a real interval (or a reunion of real intervals) $E \subset R$. Any function defined on E having complex values, $f : E \to C$, is called *a complex function of real variable*. So, for $\forall t \in R$, we have $f(t) \in C$, i.e. $f(t) = f_1(t) + i f_2(t)$, where $f_1(t)$ and $f_1(t)$ are real functions of a real variable. Also, i is the well known complex number such that $i^2 = -1$.

Due to some clear isomorphisms between R^2 and C, as linear spaces, we can identify a complex function $f(t)$ by a vectorial function (f_1, f_2), or by a function in the classical space of vectors, V_2 (for $\vec{v} \in V_2$, we have $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$).

As a consequence, the known results from R^2 with regards to the open sets, closed sets, vicinities, and so on, can be transposed, without difficulties, in the complex space C. More precisely, by introducing the following distance

$$d : C \times C \to R_+^*,\ d(z_1, z_2) = |z_1 - z_2|,\ \forall z_1, z_2 \in C,$$

and replacing $z_1 = x_1 + i y_1,\ z_2 = x_2 + i y_2$, it follows that

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

which coincides to the Euclidian distance in R^2, such that we can identify the metrics spaces (C, d) and (R^2, d).

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Let a be a fixed complex number, $a \in C$. We define the set

$$\Delta(a, \varrho) = \{z \in C : d(z, a) < \varrho\} = \{z \in C : |z - a| < \varrho\},$$

which is defined as an *open disc* of center a and radius ϱ.

The boundary of this disc is defined as $\Gamma(a, \varrho) = \{z \in C : |z - a| = \varrho\}$, such that the closed disc is

$$\overline{\Delta}(a, \varrho) = \Delta(a, \varrho) \cup \Gamma(a, \varrho).$$

The notions of limit, continuity, derivability, and so on, in the point t_0, are defined with the help of the open intervals, centred in t_0, and, respectively, relative to the set of values, with the help of the open discs, centered in a fixed point $a \in C$, with the radius ϱ.

Definition 1.1.1 The function $f : E \subset R \to C$ is a continuous function in t if and only if (by definition) the functions f_1 and f_2 are continuous functions in t, where $f(t) = f_1(t) + i f_2(t)$.

It is easy to prove that if the function f is continuous in t_0, then the function $|f|$ is continuous in t_0, too.

If t_0 is an accumulating point of $E \subset R$, then the function f has a limit in t_0 if and only if (by definition) the functions f_1 and f_2 have limits in t_0, and

$$\lim_{t \to t_0} f(t) = \lim_{t \to t_0} f_1(t) + i \lim_{t \to t_0} f_2(t).$$

Consider the real interval $I \subset R$ and define $J = I \backslash \{0\}$. Then we can define the ratio

$$\varrho(t) = \frac{f(t) - f(t_0)}{t - t_0}, \ \forall t \in J.$$

Definition 1.1.2 The function f is a derivable function in t_0 if exists and is a finite number the limit

$$\lim_{t \to t_0} \varrho(t) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

This limit is denoted by

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

The function f is a derivable function in t_0 if and only if (by definition) the functions f_1 and f_2 are derivable functions in t_0 and $f'(t_0) = f_1'(t_0) + i f_2'(t_0)$.

Definition 1.1.3 A function F is called an antiderivative of the function f on the interval $I = [a, b] \subset R$ if the derivative of F is equal to f on the interval:

$$F'(t) = f(t), \ \forall t \in [a, b].$$

It is quite obvious that if F is an antiderivative of f on $[a, b]$ then the function $F(t) + C$, where C is a constant, is also an antiderivative of f because

$$(F(t) + C)' = F'(t) + C' = F'(t) = f(t).$$

Conversely, if F and F_1 are two antiderivatives of f on $[a, b]$ their difference is necessarily equal to a constant C throughout the interval $[a, b]$:

$$F_1(t) = F(t) + C.$$

Indeed,

$$(F_1(t) - F(t))' = F_1'(t) - F'(t) = f(t) - f(t) = 0.$$

Now, remember that $(F_1(t) - F(t))' = 0$ implies that there is a constant number C such that $F_1(t) - F(t) = C$ on $[a, b]$ whence it follows $F_1(t) = F(t) + C$.

So, we have established the fact that if F is an antiderivative of f on an interval $[a, b]$, then all the possible antiderivatives of f on that interval are expressed by the formula $F(t) + C$ where any number can be substituted for C.

Theorem 1.1.1 *If $f : [a, b] \to C$ is a continuous function, then f admits an antiderivative on $[a, b]$.*

Proof We define the function

$$F : [a, b] \to C, \ F(t) = \int_a^t f(\tau)\mathrm{d}\tau.$$

So, we have

$$F(t) = \int_a^t (f_1(\tau) + i f_2(\tau))\,\mathrm{d}\tau = \int_a^t f_1(\tau)\mathrm{d}\tau + i \int_a^t f_2(\tau)\mathrm{d}\tau.$$

Because f is assumed to be continuous, we deduce that f_1 and f_2 are continuous functions. But, f_1 and f_2 are real functions such that we know for them similar results. For instance, if we denote by $F_1(t) = \int_a^t f_1(\tau)\mathrm{d}\tau$, then

$$\lim_{t \to t_0} \frac{F_1(t) - F_1(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{\int_a^t f_1(\tau)\mathrm{d}\tau - \int_a^{t_0} f_1(\tau)\mathrm{d}\tau}{t - t_0} =$$
$$= \lim_{t \to t_0} \frac{\int_{t_0}^t f_1(\tau)\mathrm{d}\tau}{t - t_0} = \lim_{t \to t_0} \frac{(t - t_0) f_1(c)}{t - t_0} = f_1(t_0),$$

where we used the mean theorem.

So, we deduce that F_1 is derivable and $F_1'(t_0) = f_1(t_0)$. It is quite obvious an analogous result for f_2, such that we have $F(t) = F_1(t) + iF_2(t)$, i.e. F is derivable and

$$F'(t) = F_1'(t) + iF_2'(t) = f_1(t) + if_2(t) = f(t),$$

and the theorem is concluded. ∎

Definition 1.1.4 Let us consider the interval $I = [a, b] \subset R$ and $f : I \to C$. We say that the function f is integrable on I if and only if (by definition) the functions f_1 and f_2 are integrable functions on I and

$$\int_a^b f(t)\mathrm{d}t = \int_a^b f_1(t)\mathrm{d}t + i\int_a^b f_2(t)\mathrm{d}t.$$

It is easy to prove the following properties of the definite integral:

$$\begin{aligned}
&\int_a^b (\alpha f(t) + \beta g(t))\,\mathrm{d}t = \alpha\int_a^b f(t)\mathrm{d}t + \beta\int_a^b g(t)\mathrm{d}t,\\
&\int_a^b f(t)\mathrm{d}t = \int_a^c f(t)\mathrm{d}t + \int_c^b f(t)\mathrm{d}t,\ c \in (a, b),\\
&\left|\int_a^b f(t)\mathrm{d}t\right| \le \int_a^b |f(t)|\mathrm{d}t.
\end{aligned}$$

Example. Let us compute the integral

$$I_1 = \int_{-\pi}^{\pi} e^{\alpha t}\cos nt\mathrm{d}t.$$

Consider the integral

$$I_2 = \int_{-\pi}^{\pi} e^{\alpha t}\sin nt\mathrm{d}t,$$

too, and then we can write

$$\begin{aligned}
I_1 + iI_2 &= \int_{-\pi}^{\pi} e^{\alpha t}(\cos nt + i\sin nt)\,\mathrm{d}t = \int_{-\pi}^{\pi} e^{\alpha t}e^{int}\mathrm{d}t =\\
= \int_{-\pi}^{\pi} e^{t(\alpha+in)}\mathrm{d}t &= \left.\frac{e^{t(\alpha+in)}}{\alpha+in}\right|_{\pi}^{\pi} = \frac{1}{\alpha+in}\left(e^{\pi(\alpha+in)} - e^{-\pi(\alpha+in)}\right) =\\
&= \frac{2}{\alpha+in}\frac{e^{\pi\alpha} - e^{-\pi\alpha}}{2}(-1)^n = (-1)^n\frac{2}{\alpha+in}\sinh\alpha\pi.
\end{aligned}$$

1.2 Complex Functions of Complex Variable

Consider the set C of all complex numbers as a metrical space endowed with the already used distance $d(z_1, z_2) = |z_1 - z_2|$. Let E be a subset of the space C.

Definition 1.2.1 Any function $f : E \to C$ is called a complex function of complex variable.

If we write the function $f(z)$ in the form $f(z) = u(x, y)+iv(x, y)$, $\forall z = x+iy \in E$, then any considerations on the complex function $f(z)$ can be reduced on the real functions $u(x, y)$ and $v(x, y)$, as follows:

(i) if z_0 is an accumulating point of E, then $f(z)$ has a limit in $z_0 \Leftrightarrow u(x, y)$ and $v(x, y)$ have limits in the point (x_0, y_0), where $z_0 = x_0 + iy_0$, and

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x, y) + i \lim_{(x,y)\to(x_0,y_0)} v(x, y).$$

Also, we have

$$\lim_{z\to z_0} |f(z)| = \left|\lim_{z\to z_0} f(z)\right|.$$

(ii) $f(z)$ is continuous in the point $z_0 \Leftrightarrow u(x, y)$ and $v(x, y)$ are continuous in the point (x_0, y_0).

Definition 1.2.2 The function $f : E \subset C \to C$ is a monogeneous function in z_0 if exists and is a finite number the following limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

We use the notation

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Proposition 1.2.1 *If f is a monogeneous function in z_0 then f is continuous in z_0.*

Proof We can write the function $f(z)$ in the form

$$f(z) = f(z_0) + \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) \Rightarrow$$

$$\lim_{z\to z_0} f(z) = f(z_0) + \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}(z - z_0) = f(z_0) + f'(z_0) \lim_{z\to z_0}(z - z_0) = f(z_0),$$

and the proposition is concluded. ■

Theorem 1.2.1 *Consider the open subset $E \subset C$, a fixed point $z_0 \in E$ and the complex function $f : E \to C$, $f(z) = u(x, y) + iv(x, y)$. If the function f is monogeneous in z_0, then the functions u and v admit partial derivatives and satisfy the conditions*

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0),$$
$$\frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0).$$

The above relations are called the Cauchy-Riemann's conditions for monogenity.

Proof Let us consider the ratio

$$\varrho(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

Since E is an open set and $z_0 \in E$, we deduce that there exists the disc

$$\Delta(z_0, r) = \{z \in C : |z - z_0| < r\} \subset E.$$

In the following we use the notations

$$\Delta_0 = \Delta \backslash \{z_0\},\ A_0 = \{z \in \Delta_0 : z = x + iy_0\},\ B_0 = \{z \in \Delta_0 : z = x_0 + iy\}$$
$$\varrho_{A_0}(z) = \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0},\ \varrho_{B_0}(z) = \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)}.$$

Since the function f is monogeneous in z_0, the limits of the last two ratios must be equal:

$$\varrho_{A_0}(z) = \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0},\ \forall z \in A_0,$$
$$\varrho_{B_0}(z) = \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i\frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)},\ \forall z \in B_0.$$

By equaliting the limits of these ratios we deduce the following relations

$$\frac{\partial u(x, y)}{\partial x}(z_0) = \frac{\partial v(x, y)}{\partial y}(z_0),$$
$$\frac{\partial v(x, y)}{\partial x}(z_0) = -\frac{\partial u(x, y)}{\partial y}(z_0),$$

i.e. the Cauchy-Riemann's conditions and the theorem is proved. ■

Remark. It is easy to see that

$$f'(z_0) = \frac{\partial u(x, y)}{\partial x}(z_0) + i\frac{\partial v(x, y)}{\partial x}(z_0) = \frac{1}{i}\left(\frac{\partial u(x, y)}{\partial y}(z_0) + i\frac{\partial v(x, y)}{\partial y}(z_0)\right).$$

Theorem 1.2.2 *Consider the open subset $E \subset C$, a fixed point $z_0 \in E$ and the complex function $f : E \to C$, $f(z) = u(x, y) + iv(x, y)$. If the functions u and v admit partial derivatives in a vicinity V of z_0 and these derivatives are continuous and satisfy the Cauchy-Riemann's conditions, then the function f is monogeneous in z_0.*

Proof Because of the hypotheses of theorem, we deduce that the functions u and v are differentiable. So, we deduce that there are functions $\alpha(z)$ and $\beta(z)$ such that

$$\lim_{z\to z_0} \alpha(z) = \lim_{z\to z_0} \beta(z) = 0,$$

and we can write

$$u(x, y) - u(x_0, y_0) = \frac{\partial u(x, y)}{\partial x}(x - x_0) + \frac{\partial u(x, y)}{\partial y}(y - y_0) + |z - z_0|\,\alpha(z),$$
$$v(x, y) - v(x_0, y_0) = \frac{\partial v(x, y)}{\partial x}(x - x_0) + \frac{\partial v(x, y)}{\partial y}(y - y_0) + |z - z_0|\,\beta(z).$$

Clearly, we have

$$\begin{aligned} f(z) - f(z_0) = u(x, y) + iv(x, y) - u(x_0, y_0) - iv(x_0, y_0) = \\ u(x, y) - u(x_0, y_0) + i\,[v(x, y) - v(x_0, y_0)] = \frac{\partial u}{\partial x}(x - x_0) + \frac{\partial u}{\partial y}(y - y_0) + \\ +i\left[\frac{\partial v}{\partial x}(x - x_0) + \frac{\partial v}{\partial y}(y - y_0)\right] + |z - z_0|\,[\alpha(z) + i\beta(z)] = \\ = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x - x_0) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)(y - y_0) + |z - z_0|\,[\alpha(z) + i\beta(z)] = \\ = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x - x_0) + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)(y - y_0) + |z - z_0|\,[\alpha(z) + i\beta(z)] = \\ = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)[x - x_0 + i\,(y - y_0)] + |z - z_0|\,[\alpha(z) + i\beta(z)]\,. \end{aligned}$$

So, we can divide by $(z - z_0)$ and obtain

$$\frac{f(z) - f(z_0)}{z - z_0} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(z) + \frac{|z - z_0|}{z - z_0}\,[\alpha(z) + i\beta(z)]\,.$$

Now, by passing to the limit, by $z \to z_0$, in the last relation we deduce that the function f is monogeneous in z_0.

Moreover, it results the formula for the derivative of f:

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(z_0) = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)(z_0).$$

The theorem is concluded. ■

Applications 1. Let us prove that the function $f : C \to C$, defined by $f(z) = \overline{z}$ has no points of monogenity.

Indeed, for $z = x + iy$ we have $f(z) = x - iy \Rightarrow u(x, y) = x$ and $v(x, y) = -y$. Then, the derivative of these functions does not satisfy the Cauchy-Riemann's conditions of monogenity, because

$$\frac{\partial u}{\partial x} = 1,\ \frac{\partial v}{\partial y} = -1 \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

2. Let us prove that the function $f : C \to C,\ f(z) = e^z$ is monogeneous in the whole complex plane C.

Indeed, for $z = x + iy$ we have $f(z) = e^x(\cos y + i \sin y) \Rightarrow u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. It is a simple exercise to verify that u and v satisfy the Cauchy-Riemann's conditions of monogenity, $\forall (x, y) \in C$.

Moreover, for the formula of derivative of f we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^z.$$

3. Let D be the set $D = \{z \in C : z = x + iy,\ x > 0\}$. For $z = x + iy$ consider the function

$$f(z) = \ln\sqrt{x^2 + y^2} + i\,\mathrm{arctg}\frac{y}{x}.$$

By simple calculations we deduce that f satisfies the Cauchy-Riemann's conditions of monogenity on D and $f'(z) = 1/z$.

Definition 1.2.3 Consider an open complex set $E \subset C$ and $f : E \to C$. The function f is called a holomorphic function on E if f is monogeneous in any point of E.

Theorem 1.2.3 *Let D be a complex domain and the function $f : \mathrm{d} \to C,\ f(z) = u(x, y) + iv(x, y)$ with $u, v \in C^2(D)$. If f is a holomorphic function on D, then the functions u and v are harmonical functions on D.*

Proof Remember that the function u is harmonical if its Laplacian is null, i.e.

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Now we use the derivative formula for the function $f(z)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$

Of course, the last equality can be written in the form

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}i + \frac{\partial v}{\partial y},$$

from where, by derivation with regard to x, then to y, it obtained the following two equalities:

$$\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y \partial x}i + \frac{\partial^2 v}{\partial x \partial y},$$
$$\frac{\partial^2 u}{\partial x \partial y} + i\frac{\partial^2 v}{\partial x \partial y} = -i\frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 v}{\partial y^2}.$$

By using the definition of two complex number, we deduce

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

relations that prove that u and v are harmonic functions and the proof of the theorem is complete. ■

Definition 1.2.4 Let E be a complex and open set and the function $f : E \to C$.

(i) A point $z_0 \in E$ is a ordinary point of the function f if there exists a disc

$$\Delta(z_0, \varrho) = \{z \in E : |z - z_0| < \varrho\},$$

such that f is a holomorphic function on Δ.

(ii) A point $z_0 \in E$ is a singular point of the function f if any disc $\Delta(z_0, \varrho)$ contains certain points where f is monogeneous and certain points where f is not monogeneous.

Definition 1.2.5 In the same conditions on the set E and the function f, a singular point $z_0 \in E$ is called an isolated singular point of the function f if there exists a disc $\Delta(z_0, \varrho)$ which does not contain any other singular point of f, excepting z_0.

Definition 1.2.6 Let D be a complex domain and the function $f : \mathrm{d} \to C$. A point $a \in D$ is called a zero for the function f if there exists a number $\alpha \in N^*$ and the function $\varphi : \mathrm{d} \to C$, φ holomorphic on D, $\varphi(a) \neq 0$, such that

$$f(z) = (z - a)^{\alpha}\varphi(z), \ \forall z \in D.$$

The number α is called the order of the zero.

Proposition 1.2.2 *The order of a zero is unique.*

Proof Suppose, by the contrary, that there exists another number $\beta \in N^*$ and the function $\psi : \mathrm{d} \to C$, ψ holomorphic on D, $\psi(a) \neq 0$ such that

$$f(z) = (z-a)^\beta \, \psi(z), \ \forall z \in D.$$

Let us suppose that $\beta > \alpha$. Then we can write

$$(z-a)^\alpha \, \varphi(z) = (z-a)^\beta \, \psi(z) \Rightarrow \varphi(z) = (z-a)^{\beta-\alpha} \, \psi(z).$$

Since the functions φ and ψ are holomorphic, it results that they are continuous and then $\lim\limits_{z\to a} \varphi(z) = \varphi(a)$ and $\lim\limits_{z\to a} \psi(z) = \psi(a)$. So, by passing to the limit, by $z \to a$ in the above relation we deduce that $\varphi(a) = 0$, that contradicts the hypothesis on φ. The proposition is concluded. ■

Theorem 1.2.4 *All zeroes of a holomorphic function are isolated.*

Proof Let a be a zero of the function f. Then there exists a number $\alpha \in N^*$ and the function $\varphi : \mathrm{d} \to C$, φ holomorphic on D, $\varphi(a) \neq 0$, such that

$$f(z) = (z-a)^\alpha \, \varphi(z), \ \forall z \in D.$$

Let us prove that there exists a disc $\Delta(a, \varrho)$ that contains no other zero of the function f, i.e. $f(z) \neq 0$, $\forall z \in \Delta(a, \varrho)\backslash\{a\}$. Suppose, ad absurdum, that $\exists z_1 \in \Delta(a, \varrho)\backslash\{a\}$ such that $f(z_1) = 0 \Leftrightarrow (z_1 - a)^\alpha \varphi(z_1) = 0 \Rightarrow \varphi(z_1) = 0$. Since φ is holomorphic on D we deduce that φ is continuous on D. Since $\varphi(a) \neq 0$ we can suppose that $|\varphi(a)| > 0$. By using the definition of continuity, we obtain: for an arbitrary $\varepsilon \in (0, |\varphi(a)|)$ $\exists \delta(\varepsilon)$ such that for $|z-a| < \delta(\varepsilon) \Rightarrow |\varphi(z) - \varphi(a)| < \varepsilon$. Consider the disc $\Delta(a, \varrho)$ where $\varrho = \delta(\varepsilon)$. If there exists $z_1 \in \Delta(a, \delta(\varepsilon))$ such that $\varphi(z_1) = 0$ then $|z_1 - a| < \delta(\varepsilon) \Rightarrow |\varphi(z_1) - \varphi(a)| < \varepsilon \Rightarrow |\varphi(a)| < \varepsilon$ and this contradicts that $\varepsilon \in (0, |\varphi(a)|)$. The theorem is concluded. ■

Definition 1.2.7 Let E be an open complex set and the function $f : E \to C$. A singular point a is called a pole for the function f if there exists a number $\alpha \in N^*$ and the function $\varphi : E \cup \{a\} \to C$, φ having a as ordinary point, $\varphi(a) \neq 0$, such that

$$f(z) = \frac{1}{(z-a)^\alpha} \varphi(z), \ \forall z \in E.$$

The number α is called the order of the pole.

Proposition 1.2.3 *The order of a pole is unique.*

Proof We can use the same procedure as in the proof of the Proposition 1.2. ■

Theorem 1.2.5 *All poles of a complex function are isolated.*

Proof We can use the same procedure as in the proof of the Theorem 1.5. ■

1.3 Elementary Complex Functions

First of all, we must state that the rules of derivation, in the case of the complex function, are the same as in the case of real functions. It is a simple exercise to prove the following rules:

$$(c.f(z))' = c.f'(z),$$

$$(f(z).g(z))' = f'(z)g(z) + f(z)g'(z),$$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)},$$

$$(fo\varphi)'(z) = f'(\varphi(z))\varphi'(z).$$

We shall call *elementary functions* some relative simple functions that are used to construct the usual functions of the concrete applications.

1. Polynomial Function. The function $P : C \to C$ is defined by

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,\ \forall z \in C.$$

Using the Cauchy-Riemann's conditions for monogenity, it is easy to prove that the function $f(z) = z = x + iy$ is holomorphic on whole C, and then by using the mathematical induction, that the function $f : C \to C$, $f(z) = z^n,\ n \in N^*$ is holomorphic on whole C, too. So, we deduce that the polynomial function is holomorphic on the whole complex plane C.

2. Rational Function. The function $R(z)$ is defined by

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomial functions.

If we denote by $E = \{z \in C : Q(z) = 0\}$, we deduce that $R(z)$ is a defined and holomorphic function on $C \setminus E$.

3. Exponential Function. This function is defined by

$$f : C \to C, \ f(z) = e^z.$$

In a previous application we already proved that this function is holomorphic on the whole plane C.

Proposition 1.3.1 *The exponential function is a periodic function with the main period $T_0 = 2\pi i$ and the general period $T_k = 2k\pi i$, $k \in Z \backslash \{0, 1\}$.*

Proof Consider a complex number $T = T_1 + iT_2$ such that $f(z+T) = f(z), \ \forall z \in C$. Since for $z = x + iy$ we have $f(z) = e^z = e^x(\cos y + i \sin y)$, it results

$$e^{x+T_1}\left[\cos(y + T_2) + i \sin(y + T_2)\right] = e^x(\cos y + i \sin y), \ \forall x, y \in R.$$

If we take the modulus in this equality, we deduce $e^{x+T_1} = e^x$ and then $T_1 = 0$. Then the previous equality leads to $\cos(y + T_2) = \cos y$ and $\sin(y + T_2) = \sin y \Rightarrow y + T_2 = y + 2k\pi \Rightarrow T_2 = 2k\pi$. The proposition is concluded. ■

4. Trigonometric and Hyperbolic Functions. These functions are defined as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \ \cosh z = \frac{e^z + e^{-z}}{2}, \ \sinh z = \frac{e^z - e^{-z}}{2}.$$

The main properties of these functions are contained in the following proposition.

Proposition 1.3.2 *The trigonometric and hyperbolic functions have the following properties:*

(1) $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$, $(\cosh z)' = \sinh z$, $(\sinh z)' = \cosh z$;
(2) $\cos z$ *and* $\sin z$ *are periodic functions with the period* $T_k = 2k\pi$
(3) $\cosh z$ *and* $\sinh z$ *are periodic functions with the period* $T_k = 2k\pi i$
(4) The relationships between the trigonometric and hyperbolic functions are: $\cos iz = \cosh z$ *and* $\sin iz = i \sinh z$.
(5) The trigonometric functions have the same zeros as the correspondent real functions and the hyperbolic functions have the following zeros:

$$\cosh z = 0 \Leftrightarrow z_k = i\left(\frac{\pi}{2} + k\pi\right), \ \sinh z = 0 \Leftrightarrow z_k = k\pi i.$$

Proof (1) By direct calculations:

$$(\cos z)' = \left(\frac{e^{iz} + e^{-iz}}{2}\right)' = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right)' = \frac{i}{2}\left(e^{iz} - e^{-iz}\right)' = -\sin z.$$

(2) By using the definition of a periodic function, we have

$$\cos(z+T) = \cos z \Leftrightarrow e^{i(z+T)} + e^{-i(z+T)} = e^{iz} + e^{-iz}.$$

We now multiply both sides of the last equality by e^{iT}:

$$e^{i(z+T)}e^{iT} + e^{-iz} = e^{i(z+T)} + e^{-iz}e^{iT} \Rightarrow \left(e^{iT} - 1\right)e^{i(z+T)} - \left(e^{iT} - 1\right)e^{-iz} = 0 \Rightarrow$$
$$\Rightarrow \left(e^{iT} - 1\right)\left(e^{i(z+T)} - e^{-iz}\right) = 0 \Rightarrow e^{i(z+T)} = e^{-iz} \text{ or } e^{iT} = 1.$$

If the first conclusion is true we deduce that $e^{2iz} = e^{-iT}$, $\forall z \in C$, i.e. the exponential function is constant, that is false. Bu using the second conclusion we deduce that $\cos T + i \sin T = 1$, then $T_k = 2k\pi$.

(3) By using the definition of a periodic function, we have

$$\cosh(z+T) = \cos z \Leftrightarrow e^{z+T} + e^{-(z+T)} = e^{z} + e^{-z}.$$

We now multiply both sides of the last equality by e^{T}:

$$e^{z+T}e^{T} + e^{-z} = e^{z+T} + e^{-z}e^{T} \Rightarrow \left(e^{T} - 1\right)e^{z+T} - \left(e^{iT} - 1\right)e^{-z} = 0 \Rightarrow$$
$$\Rightarrow \left(e^{T} - 1\right)\left(e^{z+T} - e^{-z}\right) = 0 \Rightarrow e^{z+T} = e^{-z} \text{ or } e^{T} = 1.$$

If the first conclusion is true we deduce that $e^{2z} = e^{-T}$, $\forall z \in C$, i.e. the exponential function is constant, that is false. Bu using the second conclusion we deduce that

$$e^{T_1}\left(\cos T + i \sin T\right) = 1,$$

for $T = T_1 + iT_2$. By passing to the modulus we obtain $e^{T_1} = 1$ such that $T_1 = 0$. Also, from above, using the fact that $T_1 = 0$, it results $\cos T + i \sin T = 1$, such that $\cos T_2 = 1$ and $\sin T_2 = 0$, i.e. $T_2 = 2k\pi$. Finally, $T = 2k\pi i$.

(4) We have

$$\cos iz = \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z,$$

$$\sin iz = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = \frac{e^{-z} - e^{z}}{2i} = i\frac{e^{z} - e^{-z}}{2} = i \sinh z.$$

(5) The zeros of the $\cos z$ are the solution of the equation $\cos z = 0$. Then

$$e^{iz} + e^{-iz} = 0 \Rightarrow e^{2iz} = -1 \Rightarrow e^{2i(x+iy)} = -1 \Rightarrow e^{-2y}(\cos 2x + i \sin 2x) = -1$$

By passing to the modulus we obtain

$$e^{-2y} = 1 \Rightarrow y = 0,$$

and then, from the last equality, it results

$$\cos 2x + i \sin 2x = -1 \Rightarrow \cos 2x = -1, \sin 2x = 0 \Rightarrow$$

$$\Rightarrow 2x = (2k+1)\pi \Rightarrow x = \frac{\pi}{2} + k\pi, k \in Z.$$

In a similar manner we obtain that for the zeros of the function $\sin z$ we have $y = 0$ and $x = k\pi, k \in Z$, i.e. $z_k = k\pi, k \in Z$.
Similarly,

$$\cosh z = 0 \Leftrightarrow z_k = i\left(\frac{\pi}{2} + k\pi\right), \ \sinh z = 0 \Leftrightarrow z_k = k\pi i, k \in Z,$$

and the proposition is concluded.

■

It is a simple exercise to prove the following relations:

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2,$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$

$$\sin 2z = 2 \sin z \cos z, \ \cos 2z = \cos^z - \sin^2 z, \ \sin^2 z + \cos^2 z = 1,$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cos z_2 \pm \sinh z_1 \sinh z_2,$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2,$$

$$\sinh 2z = 2 \sinh z \cosh z, \ \cosh 2z = \cos^z + \sin^2 z, \ \cosh^2 z - \sinh^2 z = 1.$$

5. Radical Function. Let us consider the function $f(z) = \sqrt{z}$. Remember that we can write z in the form $z = \varrho(\cos\theta + i \sin\theta) = \varrho e^{i\theta}$ and then we have

$$\sqrt{z} = \pm\varrho^{1/2} e^{i\theta/2}.$$

So, for one z we find two values for the radical function and we have the functions

$$f_1(z) = \varrho^{1/2} e^{i\theta/2}, \ f_2(z) = -\varrho^{1/2} e^{i\theta/2}.$$

Definition 1.3.1 Function f is called a multi-form if almost two values of $f(z)$ correspond to a single z.

If z is placed on a curve that does not contain the origin, then the functions f_1 and f_2 have the above values. But, if z is placed on a closed curve that contains the origin, then $\arg z$ increases by 2π when z moves along this curve, starting from a point M, which belongs to this curve, and arrives again in M. Hence,

$$f_1(z) = \varrho^{1/2} e^{i(\theta+2\pi)/2} = \varrho^{1/2} e^{i\theta/2} e^{i\pi} = -\varrho^{1/2} \varrho^{i\theta/2} = f_2.$$

In the same manner we obtain $f_2(z) = f_1(z)$. Hence, the two branches of the radical function pass one to one when z moves along a closed curve that contains the origin.

We say the $z = 0$ is *the critical point* or *the point of ramification* for the radical function. To make the branches f_1 and f_2 as uniform functions we must make a cut in the complex plane through a semi-axis starting from the origin. So, the point z cannot move on a closed curve that contains the origin.

Another procedure to make the branches f_1 and f_2 as uniform functions consists in a superposition of two identical complex planes, making a cut along a semi-axix starting from the origin. This manner is known as the *Riemannian surfaces* method. Analogical considerations are still valid in the case of the more general radical function $f(z) = \sqrt[m]{z}$, but in this case it is necessary to consider m Riemannian surfaces.

6. Logarithmic Function. This function is denoted by Lnz and is defined as the solution of the equation

$$e^{f(z)} = z.$$

If we denote $f(z) = u + iv$ and $z = \varrho(\cos\theta + i\sin\theta)$ then

$$e^u(\cos v + i\sin v) = \varrho(\cos\theta + i\sin\theta) \Rightarrow e^u = \varrho,\ v = \theta + 2k\pi, k \in Z,$$

$$u = \ln\varrho,\ v = \theta + 2k\pi, k \in Z \Rightarrow \ \mathrm{Ln}z = \ln|z| + i(\theta + 2k\pi), k \in Z,$$

where θ is the main argument of z and is denoted by $\theta =$arg$z$. So, we can write the logarithmic function in the form

$$\mathrm{Ln}z = \ln|z| + i(\arg z + 2k\pi).$$

If we denote by $\ln z = \ln|z| + i\arg z$, that is called *the main value* of the logarithmic function, we can write

$$\mathrm{Ln}z = \ln z + 2k\pi i, k \in Z.$$

So, it is easy to see that the logarithmic function is a multi-form function with an infinite number of branches:

$$\begin{aligned} f_0(z) &= \ln z, \\ f_1(z) &= \ln z + 2\pi i, \\ f_2(z) &= \ln z + -\pi i, \\ &\vdots \end{aligned}$$

If z is placed on a closed curve with the origin inside, the $f_0(z)$ becomes $f_1(z)$, $f_1(z)$ becomes $f_2(z)$, and so on. So, the branches of a logarithm function are multi-form functions, that become uniform functions with the help of a cut in the complex

plane along of a semi-axis that starts from the origin. The point $z = 0$ is called *the point of ramification* for the logarithmic function.

The main properties of the logarithmic function are included in the following proposition.

Proposition 1.3.3 *For the main value of the Logarithmic Function we have:*

$$\begin{aligned} \ln(z_1 z_2) &= \ln z_1 + \ln z_2, \\ \ln\frac{z_1}{z_2} &= \ln z_1 - \ln z_2, \\ \ln z^n &= n \ln z, \\ \ln \sqrt[n]{z} &= \frac{1}{n} \ln z. \end{aligned}$$

Proof All these relations can be proved by using the formula of logarithmic function. For instance,

$$\ln(z_1 z_2) \ln|z_1 z_2| + i \arg(z_1 z_2) = \ln|z_1| + \ln|z_2| + i\,(\arg z_1 + \arg z_2)\,.$$

The same procedure can be used for other properties. ■

Remark. The above properties are still not valid for $\mathrm{Ln} z$. For instance

$$\mathrm{Ln}(z_1 z_2) = \ln z_1 + \mathrm{Ln} z_2 = \mathrm{Ln} z_1 + \ln z_2.$$

7. Power Function. The power function is defined by the formula

$$f(z) = z^\alpha = e^{\alpha \mathrm{Ln} z}.$$

Of course, because of the logarithmic function, the power function is a multi-form function.

Proposition 1.3.4 *If $\alpha \notin Q$ then the function $f(z) = z^\alpha$ has an infinite number of branches and if $\alpha \in Q$ then the power function has a finite number of branches.*

Proof Of course, we can write the power function in the form

$$f(z) = z^\alpha = e^{\alpha \mathrm{Ln} z} = e^{\alpha[\ln \varrho + i(\theta + 2k\pi)]}, \ k \in Z.$$

In order to have a finite number of branches, there must exist two different values of k which give the same value for z^α.

$$e^{\alpha[\ln \varrho + i(\theta + 2k_1\pi)]} = e^{\alpha[\ln \varrho + i(\theta + 2k_2\pi)]} \Rightarrow e^{2k_1\pi\alpha} = e^{2k_2\pi\alpha} \Rightarrow e^{2\pi\alpha i(k_1 - k_2)} = 1 \Rightarrow$$

$$\Rightarrow 2\pi\alpha i(k_1 - k_2) = 2m\pi \Rightarrow (k_1 - k_2)\alpha = m \Rightarrow \alpha = \frac{m}{k_1 - k_2}, \ m, k_1, k_2 \in Z,$$

and the proposition is concluded. ■

Application. We want to prove that the expression i^i has real values only. Indeed,

$$i^i = e^{i\mathrm{Ln} i} = e^{i(\ln|i|+i(\arg(i)+2k\pi)} = e^{-(\pi/2+2k\pi)},\ k \in Z.$$

8. Inverse Trigonometric Functions. By definition, the inverse function of the function $\cos z$ is the solution of the equation $\cos f(z) = z$ and is denoted by $\mathrm{Arccos} z$. In the following we want to give an explicit expression for the function $\mathrm{Arccos} z$.

$$e^{if} + e^{-if} = 2z \Rightarrow e^{2if} - 2ze^{if} + 1 = 0; \Delta = z^2 - 1 = i^2(1 - z^2) \Rightarrow$$

$$\Rightarrow e^{if} = z \pm i\sqrt{1 - z^2} \Rightarrow if = \mathrm{Ln}\left(z \pm i\sqrt{1 - z^2}\right) \Rightarrow$$

$$\Rightarrow \mathrm{Arccos} z = \frac{1}{i}\mathrm{Ln}\left(z \pm i\sqrt{1 - z^2}\right).$$

The quantity $1/i \ln\left(z + i\sqrt{1 - z^2}\right)$ is called the main part of the function $\mathrm{Arccos} z$ and is denoted by $\arccos z$. For the other part, we have

$$\ln\left(z - i\sqrt{1 - z^2}\right) = \ln\frac{z^2 - i^2(1 - z^2)}{z + i\sqrt{1 - z^2}} = \ln\frac{1}{z + i\sqrt{1 - z^2}} = -\ln(z + i\sqrt{1 - z^2}).$$

Hence,

$$\mathrm{Arccos} z = \frac{1}{i}\left[\ln\left(z \pm i\sqrt{1 - z^2}\right) + 2k\pi\right] =$$

$$= 2k\pi + \frac{1}{i}\ln\left(z \pm i\sqrt{1 - z^2}\right) = 2k\pi \pm \arccos z.$$

By definition, the inverse function of the function $\sin z$ is the solution of the equation $\sin f(z) = z$ and is denoted by $\mathrm{Arcsin} z$. In the following we want to give an explicit expression for the function $\mathrm{Arcsin} z$.

$$e^{if} - e^{-if} = 2iz \Rightarrow e^{2if} - 2ize^{if} - 1 = 0; \Delta = i^2z^2 + 1 = 1 - z^2 \Rightarrow$$

$$\Rightarrow e^{if} = iz \pm \sqrt{1 - z^2} \Rightarrow if = \mathrm{Ln}\left(iz \pm \sqrt{1 - z^2}\right) \Rightarrow$$

$$\Rightarrow \mathrm{Arcsin} z = \frac{1}{i}\mathrm{Ln}\left(iz \pm \sqrt{1 - z^2}\right).$$

The quantity $1/i \ln\left(iz + \sqrt{1 - z^2}\right)$ is called the main part of the function $\mathrm{Arccos} z$ and is denoted by $\arcsin z$. For the other part, we have

$$\ln\left(iz - \sqrt{1 - z^2}\right) = \ln\frac{i^2z^2 - 1 + z^2}{iz + \sqrt{1 - z^2}} = \ln\frac{-1}{iz + \sqrt{1 - z^2}}$$

$$= \ln(-1) - \ln(iz + \sqrt{1-z^2}) = \ln(|-1|) + i\arg(-1) - \ln(iz + \sqrt{1-z^2}) =$$

$$= \pi i - \ln(iz + \sqrt{1-z^2}).$$

Hence,

$$\text{Arcsin}z = \frac{1}{i}\left[\ln\left(iz + \sqrt{1-z^2}\right) + 2k\pi\right] = 2k\pi + \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right) \Rightarrow$$

$$\Rightarrow \text{Arcsin}z = 2k\pi + \arcsin z.$$

$$\text{Arcsin}z = \frac{1}{i}\left[\ln\left(iz - \sqrt{1-z^2}\right) + 2k\pi\right] = 2k\pi + \frac{1}{i}\left[\pi i - \ln\left(iz + \sqrt{1-z^2}\right)\right] =$$

$$= 2k\pi + \pi - \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right) \Rightarrow$$

$$\Rightarrow \text{Arcsin}z = (2k+1)\pi - \arcsin z.$$

1.4 Complex Integrals

Consider $\gamma : [a, b] \to C,\ [a, b] \subset R$, a parametrized and smooth (i.e. $\gamma \in C^1[a, b]$) curve. Let f be a continuous function $f : E \to C$, where E is an open subset of C. Then $(f o\gamma)(t)\gamma'(t)$ is a continuous function, too, and, hence there exists the integral

$$\int_a^b (f o\gamma)(t)\gamma'(t)\mathrm{d}t.$$

Definition 1.4.1 In the above conditions imposed on the curve γ and the function f we define the complex integral of the function f on the curve γ by

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b (f o\gamma)(t)\gamma'(t)\mathrm{d}t.$$

Remarks.

(1) If we write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ then $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ and

$$f(z)\mathrm{d}z = (u + iv)(\mathrm{d}x + i\mathrm{d}y) = u\mathrm{d}x - v\mathrm{d}y + i(u\mathrm{d}y + v\mathrm{d}x),$$

hence,

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma [u(x, y)\mathrm{d}x - v(x, y)\mathrm{d}y] + i \int_\gamma [u(x, y)\mathrm{d}y + v(x, y)\mathrm{d}x].$$

(2) If we write $\gamma(t) = \alpha(t) + i\beta(t)$ then $\gamma'(t) = \alpha'(t) + i\beta'(t)$. So,

$$f(\gamma(t))\gamma'(t) = [u(\alpha, \beta) + iv(\alpha, \beta)] \left(\alpha' + i\beta'\right) =$$

$$= u(\alpha, \beta)\alpha' - v(\alpha, \beta)\beta' + i \left[u(\alpha, \beta)\beta' + v(\alpha, \beta)\alpha'\right].$$

Then the complex integral becomes

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma f(\gamma(t))\gamma'(t)\mathrm{d}t =$$

$$= \int_\gamma \left[u(\alpha, \beta)\alpha' - v(\alpha, \beta)\beta'\right] \mathrm{d}t + i \int_\gamma \left[u(\alpha, \beta)\beta' + v(\alpha, \beta)\alpha'\right] \mathrm{d}t.$$

The main properties of the complex integral are contained in the next proposition.

Proposition 1.4.1 *(i) For any two parametrized, smooth and equivalent curves, γ_1 and γ_2, we have*

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z.$$

(ii) For any two complex and continuous functions f and g and any two complex constants α and β we have

$$\int_\gamma (\alpha f(z) + \beta g(z))\mathrm{d}z = \alpha \int_\gamma f(z)\mathrm{d}z + \beta \int_\gamma g(z)\mathrm{d}z.$$

(iii) When a point moves on the curve γ in the contrary sense as the positive sense (established by convention) then

$$\int_{\gamma^-} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z.$$

Proof It is easy to prove these assertions by means of similar properties of the real integral. ∎

Application. Let us compute the integral

$$I_n = \int\limits_{\gamma} (z-a)^n \mathrm{d}z,$$

where γ is the circle having the origin as center and the radius equal to r. We can write $\gamma(t) = a + re^{it},\ t \in [0, 2\pi]$, $\gamma'(t) = rie^{it}$,

$$\int\limits_{\gamma} f(z)\mathrm{d}z = \int\limits_{0}^{2\pi} r^n e^{int} rie^{it}\mathrm{d}t = r^{n+1} i \int\limits_{0}^{2\pi} e^{it(n+1)}\mathrm{d}t.$$

Then, for $n = -1 \Rightarrow I_{-1} = 2\pi i$, and for $n \neq -1$ we have

$$I_n = r^{n+1} i \left.\frac{e^{it(n+1)}}{i(n+1)}\right|_0^{2\pi} = \frac{r^{n+1}}{n+1}\left[e^{2i(n+1)\pi} - 1\right] = 0.$$

Definition 1.4.2 If $\gamma : [a, b] \to E \subset C$ is a piecewise smooth curve (i.e. γ is a reunion of a finite number, m, of smooth curves, then the complex integral of the complex function f on γ is defined as

$$\int\limits_{\gamma} f(z)\mathrm{d}z = \sum_{k=1}^{m} \int\limits_{\gamma_k} f(z)\mathrm{d}z.$$

Application. Let us compute the integral

$$I = \int\limits_{\gamma} z^n \mathrm{d}z,\ \ n \in N.$$

Here $\gamma = \gamma_1 \cup \gamma_2$, where γ_2 is the semi-circle having the origin as center and the radius equal to a and γ_1 is the interval $[-a, a]$. We can write $\gamma_1(t) = t,\ t \in [-a, a]$ and $\gamma_2(t) = ae^{it},\ t \in [0, \pi]$. Then

$$I = \int\limits_{\gamma} f(z)\mathrm{d}z = \int\limits_{-a}^{a} t^n \mathrm{d}t + \int\limits_{0}^{\pi} e^{itn} a^n aie^{it}\mathrm{d}t.$$

By simple calculations we obtain that $I = 0$.

In the next theorem we prove a very important result of the theory of complex functions. The result is due to Cauchy.

Theorem 1.4.1 *Let Δ be a bounded domain in the complex plane C having the boundary Γ, where Γ is a reunion of a finite number of closed, simple and smooth curves. If f is a holomorphic function on $\overline{\Delta} = \Delta \cup \Gamma$ then*

$$\int_{\Gamma} f(z)\mathrm{d}z = 0.$$

Proof Without loss of generality, we suppose that $f \in C^1(\overline{\Delta})$ and that means $u, v \in C^1(\overline{\Delta})$. We now remember the Riemann-Green's formula. If $P(x, y)$ and $Q(x, y)$ are functions of class $C^1(\overline{\Delta})$ then

$$\int_{\partial\Delta} P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = \iint_{\Delta} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y.$$

Based on the hypothesis, f is a holomorphic function and that means that the Cauchy-Riemann's relations are still valid:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Based on these relations and the Riemann-Green formula we have

$$\int_{\Gamma} f(z)\mathrm{d}z = \iint_{\Delta} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y + \iint_{\Delta} \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\mathrm{d}x\mathrm{d}y = 0.$$

The theorem is concluded. ■

Corollary 1.4.1 *Let D be a simple conex domain and γ a simple, closed and smooth curve included in D, $\gamma \subset D$. If f is a holomorphic function on D, then*

$$\int_{\gamma} f(z)\mathrm{d}z = 0.$$

Proof Since $\gamma \subset D$ we deduce that Δ, which is the domain closed by γ, is included in D. Based on Theorem 1.4.1 we deduce that the integral on the boundary of Δ is null. But the boundary of Δ is γ such that

$$\int_{\gamma} f(z)\mathrm{d}z = 0.$$

Corollary 1.4.2 *Let D be a multi-conex domain having m voids, $\Delta_1, \Delta_2, \ldots, \Delta_m$, bounded by $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ which are simple, closed and smooth curves included in D. If Γ_0 is the outside boundary of D, then*

$$\int_{\Gamma_0} f(z)\mathrm{d}z = \sum_{k=1}^{m} \int_{\Gamma_k} f(z)\mathrm{d}z.$$

Proof. Consider

$$\Delta = D \backslash \left(\overline{\Delta_1} \cup \overline{\Delta_2} \cup \ldots \cup \overline{\Delta_m} \right).$$

The boundary of Δ is $\Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_m = \Gamma$. Because the voids were avoided, we deduce that f is holomorphic on Δ and then, based on the Theorem 1.4.1, we deduce

$$\int_{\partial\overline{\Delta}} f(z)\mathrm{d}z = \int_{\Gamma} f(z)\mathrm{d}z = 0.$$

But the sense on the curves Γ_k is negative and then

$$\int_{\Gamma_0} f(z)\mathrm{d}z + \int_{\Gamma_1^-} f(z)\mathrm{d}z + \int_{\Gamma_2^-} f(z)\mathrm{d}z + \cdots + \int_{\Gamma_m^-} f(z)\mathrm{d}z = 0.$$

Since

$$\int_{\Gamma_k^-} f(z)\mathrm{d}z = -\int_{\Gamma_k} f(z)\mathrm{d}z,$$

we deduce

$$\int_{\Gamma_0} f(z)\mathrm{d}z = \sum_{k=1}^{m} \int_{\Gamma_k} f(z)\mathrm{d}z.$$

Application. Let us compute the integral

$$\int_{\Gamma} \frac{z}{z^2-1}\mathrm{d}z,$$

where Γ is a circle having the origin as center and the radius equal to $a \neq 1$.

(1) If $a < 1$ then the domain bounded by Γ is simple conex and f is holomorphic such that

$$\int_{\Gamma} \frac{z}{z^2-1}\mathrm{d}z = 0,$$

based on Theorem 1.4.1

(2) If $a > 1$ the domain Δ bounded by Γ is double conex. Consider the domain

$$\Delta_0 = \Delta \backslash \left(\overline{\Delta}_1 \cup \overline{\Delta}_2 \right),$$

where the points -1 and 1 was isolated by two circle Γ_1 and Γ_2, respectively.

Because f is holomorphic on Δ_0 we deduce that

$$\int_{\partial\Delta_0} f(z)\mathrm{d}z = 0.$$

But the boundary of Δ_0 is

$$\partial\Delta_0 = \Gamma \cup \Gamma_1^- \cup \Gamma_2^-.$$

By using corollary 1.2 we obtain

$$\int_{\Gamma} \frac{z}{z^2-1}\mathrm{d}z = \int_{\Gamma_1} \frac{z}{z^2-1}\mathrm{d}z + \int_{\Gamma_2} \frac{z}{z^2-1}\mathrm{d}z = I_1 + I_2.$$

If we write

$$\frac{z}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1} \right),$$

then

$$I_1 = \frac{1}{2}\left[\int_{\Gamma_1} \frac{1}{z-1}\mathrm{d}z + \int_{\Gamma_1} \frac{1}{z+1}\mathrm{d}z \right] = \frac{1}{2}(0 + 2\pi i) = \pi,$$

$$I_2 = \frac{1}{2}\left[\int_{\Gamma_2} \frac{1}{z-1}\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z+1}\mathrm{d}z \right] = \frac{1}{2}(2\pi i + 0) = \pi.$$

Finally, for $a > 1$ we have $I = 2\pi i$.

Theorem 1.4.2 *Let D be a simple conex domain in the complex plane C and f : d $\to C$ a holomorphic function. Consider L_1 and L_2 two simple and smooth curves included in D and having the same extremities. Then*

$$\int_{L_1} f(z)\mathrm{d}z = \int_{L_2} f(z)\mathrm{d}z.$$

Proof Firstly, consider the case when the curves L_1 and L_2 have no other common points, without the extremities. Then $L_1 \cup L_2$ is a closed, simple and smooth curve

that closes the domain $\Delta \subset D$. If the sense on L_1 is positive, on L_2 the sense is negative. Based in corollary 1.4.1 we have

$$\int_{\partial\Delta} f(z)\mathrm{d}z = 0.$$

But $\partial\Delta = L_1 \cup L_2^-$ and then

$$\int_{\partial\Delta} f(z)\mathrm{d}z = \int_{L_1} f(z)\mathrm{d}z + \int_{L_2^-} f(z)\mathrm{d}z = 0 \Rightarrow$$

$$\Rightarrow \int_{L_1} f(z)\mathrm{d}z = \int_{L_2} f(z)\mathrm{d}z.$$

Consider now the case when the curves L_1 and L_2 have other common points. We can take another curve L_3 that has no other common points both on L_1 and L_2. By using the first part of the proof, we deduce

$$\int_{L_1} f(z)\mathrm{d}z = \int_{L_3} f(z)\mathrm{d}z, \int_{L_2} f(z)\mathrm{d}z = \int_{L_3} f(z)\mathrm{d}z \Rightarrow$$

$$\Rightarrow \int_{L_1} f(z)\mathrm{d}z = \int_{L_2} f(z)\mathrm{d}z.$$

The theorem is proved. ∎

Remark. The previous theorem afirms that in a simple conex domain the integral of a holomorphic function is independent of the curve that lies between two complex points z_0 and z_1.

Theorem 1.4.3 *Let D be a simple conex domain in the complex plane C and f : d $\to C$ a holomorphic function. For a fixed point $z_0 \in D$ we define the function*

$$F(z) = \int_{z_0}^{z} f(t)\mathrm{d}t, \ \forall t \in D.$$

Then the function F is holomorphic on D and $F'(z) = f(z), \ \forall z \in D$.

Proof As we already know

$$\int_{z_0}^{z} f(t)\mathrm{d}t = \int_{z_0}^{z} u\mathrm{d}x - v\mathrm{d}y + i\int_{z_0}^{z} u\mathrm{d}y + v\mathrm{d}x.$$

Based on the previous theorem the last two integrals are independent of the curves that lie between the points z_0 and z and then $\exists U(x, y),\ V(x, y)$ such that $\mathrm{d}U = u\mathrm{d}x - v\mathrm{d}y$ and $\mathrm{d}V = u\mathrm{d}y + v\mathrm{d}x$. So,

$$F(z) = \int_{z_0}^{z} \mathrm{d}U + i \int_{z_0}^{z} \mathrm{d}V = U + iV.$$

But

$$\frac{\partial U}{\partial x} = u,\ \frac{\partial U}{\partial y} = -v,\ \frac{\partial V}{\partial x} = v,\ \frac{\partial V}{\partial x} = u, \Rightarrow$$

$$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial V}{\partial x},\ \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x},$$

i.e. U and V satisfy the Cauchy-Riemann's conditions. So, we deduce that the function F is holomorphic on D and

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

The theorem is concluded. ■

Definition 1.4.3 Let D be a simple conex domain in the complex plane C and the function $f : D \to C$. A function $F : D \to C$ is an antiderivative of f if:

(1) F is a holomorphic function on D;
(2) $F'(z) = f(z),\ \forall z \in D$.

Remark. The function F defined in Theorem 1.4.3 is an antiderivative of the function f.

It is easy to prove the following two properties of the antiderivates, included in the next proposition.

Proposition 1.4.2 *For a given complex function f we have:*

(1) If F is an antiderivative of f then $F+K$ is an antiderivative too, for any complex constant K.
(2) If F_1 and F_2 are two antiderivatives of the function f then $F_1 - F_2$ is constant.

Theorem 1.4.4 (Cauchy's Formula) *Let D be a bounded domain in the complex plane C with boundary Γ which is a reunion of a finite number of closed, simple and smooth curves. If f is a holomorphic function and we know the values of f on the boundary, then we can compute the values of f in any point of D by using the formula*

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - a} \mathrm{d}z,\ \forall a \in D.$$

Proof Let a be an arbitrary fixed point, $a \in D$. Define the function

$$g(z) = \frac{f(z)}{z-a},$$

which is holomorphic on $D \backslash \{a\}$.

Consider a disc ω having a as center and a sufficient small radius such that $\omega \subset D$. Denote by γ the boundary of the disc ω and $\Delta = D \backslash (\omega \cup \gamma)$. Of course, g is holomorphic on Δ and, based on the Cauchy's theorem we have $\int\limits_{\partial\Delta} g(z) = 0$, where $\partial\Delta = \Gamma \cup \gamma^-$. Hence, we have

$$\int\limits_{\Gamma} g(z)\mathrm{d}z + \int\limits_{\gamma^-} g(z)\mathrm{d}z = 0 \Rightarrow \int\limits_{\Gamma} g(z)\mathrm{d}z = \int\limits_{\gamma} g(z)\mathrm{d}z \Rightarrow$$

$$\Rightarrow \int\limits_{\Gamma} \frac{f(z)}{z-a}\mathrm{d}z = \int\limits_{\gamma} \frac{f(z)}{z-a}\mathrm{d}z = \int\limits_{\gamma} \frac{f(z)-f(a)+f(a)}{z-a}\mathrm{d}z \Rightarrow$$

$$\Rightarrow \int\limits_{\Gamma} \frac{f(z)}{z-a}\mathrm{d}z = \int\limits_{\gamma} \frac{f(z)-f(a)}{z-a}\mathrm{d}z + f(a)\int\limits_{\gamma} \frac{1}{z-a}\mathrm{d}z. \tag{1.4.1}$$

Based on a previous application, we know that

$$\int\limits_{\gamma} \frac{1}{z-a}\mathrm{d}z = 2\pi i,$$

because γ is a circle having a as center.

To arrive at the desired result we must prove that

$$\int\limits_{\gamma} \frac{f(z)-f(a)}{z-a}\mathrm{d}z = 0.$$

Since f is holomorphic, we deduce that f is continuous and then $\forall\varepsilon > 0\ \exists\delta(\varepsilon)$ such that $\forall z \in \Delta,\ |z-a| < \delta(\varepsilon)$ we have $|f(z) - f(a)| < \varepsilon$. But we can chose the radius of the circle γ such that $\varrho < \delta(\varepsilon)$ and then

$$\left|\frac{f(z)-f(a)}{z-a}\right| = \frac{|f(z)-f(a)|}{|z-a|} = \frac{|f(z)-f(a)|}{\varrho} < \frac{\varepsilon}{\varrho}.$$

Therefore,

$$\left|\int_\gamma \frac{f(z)-f(a)}{z-a}\mathrm{d}z\right| \le \int_\gamma \left|\frac{f(z)-f(a)}{z-a}\right| \mathrm{d}z < \int_\gamma \frac{\varepsilon}{\varrho}\mathrm{d}z = \frac{\varepsilon}{\varrho}2\pi\varrho = 2\pi\varepsilon.$$

Then, the formula (4.1) becomes

$$\int_\Gamma \frac{f(z)}{z-a}\mathrm{d}z = 2\pi f(a),$$

and the theorem is proved. ■

Remark. In the case when the domain is multi-conex, the boundary is given by

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_m,$$

the Cauchy's formula becomes

$$f(a) = \frac{1}{2\pi i}\int_{\Gamma_0} \frac{f(z)}{z-a}\mathrm{d}z - \frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\Gamma_k} \frac{f(z)}{z-a}\mathrm{d}z, \ \forall a \in D.$$

Application. Let us compute the integral

$$I = \int_\gamma \frac{\cos \pi z}{z^2+1}\mathrm{d}z,$$

where γ is a simple, closed and smooth curve that does not pass through -1 and i. Denote by D the domain closed by γ.

(1) If $-i, i \notin D$, since f is a holomorphic function we obtain that $I = 0$.
(2) If $-i \in D$ we can write

$$I = \int_\gamma \frac{\varphi(z)}{z+i}\mathrm{d}z, \ \varphi(z) = \frac{\cos \pi z}{z-i},$$

such that, based on the Cauchy's formula we have

$$I = 2\pi i\varphi(-i) = -\pi \cos \pi i = -\pi \cosh \pi.$$

(3) If $i \in D$ we can write

$$I = \int_{\gamma} \frac{\varphi(z)}{z - i} \mathrm{d}z, \ \varphi(z) = \frac{\cos \pi z}{z + i},$$

such that, based on the Cauchy's formula we have

$$I = 2\pi i \varphi(i) = \pi \cos \pi i = \pi \cosh \pi.$$

(4) If $-i, i \in D$ we take two circles γ_1, γ_2 having the centers $-i$ and i, respectively and then we can write

$$I = \int_{\gamma} f(z)\mathrm{d}z = I = \int_{\gamma_1} \frac{\varphi(z)}{z + i}\mathrm{d}z + \int_{\gamma_2} \frac{\psi(z)}{z - i}\mathrm{d}z, \ \varphi(z) = \frac{\cos \pi z}{z - i}, \ \psi(z) = \frac{\cos \pi z}{z + i}.$$

Theorem 1.4.5 *Let E be an open complex set. Consider φ a continuous function, $\varphi : E \to C$ and Γ a simple and smooth curve contained in E. Then the function*

$$f_n(z) = \int_{\Gamma} \frac{\varphi(\tau)}{(\tau - z)^n} \mathrm{d}z,$$

is a holomorphic function on $C \backslash \Gamma$ and its derivative has the form

$$f_n'(z) = n \int_{\Gamma} \frac{\varphi(\tau)}{(\tau - z)^{n+1}} \mathrm{d}z.$$

Proof To speed the proof we consider only the case $n = 1$. Of course, the function $\varphi(\tau)/(\tau - z)$ is defined and continuous for any $\tau \in E \backslash \{z\}$. Consider a disc ω bounded by the circle γ having z as the center and the radius ϱ is sufficient small such that $\omega \cup \gamma \subset E \backslash \Gamma$. We must prove that there exists and is a finite number the following limit

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{\Gamma} \varphi(\tau) \left(\frac{1}{\tau - z - h} - \frac{1}{\tau - z} \right) \mathrm{d}\tau =$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{\Gamma} \varphi(\tau) \frac{h}{(\tau - z - h)(\tau - z)} \mathrm{d}\tau = \lim_{h \to 0} \int_{\Gamma} \varphi(\tau) \frac{1}{(\tau - z - h)(\tau - z)} \mathrm{d}\tau.$$

We can write

$$\frac{1}{(\tau - z - h)(\tau - z)} = \frac{1}{(\tau - z)^2} + \frac{h}{(\tau - z - h)(\tau - z)^2},$$

and the previous limit becomes

$$\lim_{h\to 0}\left[\int_\Gamma \frac{\varphi(\tau)}{(\tau-z)^2}\mathrm{d}\tau + h\int_\Gamma \frac{\varphi(\tau)}{(\tau-z)^2(\tau-z-h)}\mathrm{d}\tau\right].$$

Since φ is a continuous function we have

$$\exists M = \sup_{\tau\in\Gamma}|\varphi(\tau)|.$$

Also,

$$|\tau - z| > \varrho,\ |\tau - z - h| > |\tau - z| - |h| > \varrho - |h|.$$

Then

$$\left|\int_\Gamma \frac{\varphi(\tau)}{(\tau-z)^2(\tau-z-h)}\mathrm{d}\tau\right| \le \int_\Gamma \left|\frac{\varphi(\tau)}{(\tau-z)^2(\tau-z-h)}\right|\mathrm{d}\tau < \frac{M}{\varrho^2(\varrho-|h|)}L(\Gamma).$$

Hence, the last integral is finite. Here $L(\Gamma)$ is the measure of the arc Γ. Finally, we obtain

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \int_\Gamma \frac{\varphi(\tau)}{(\tau-z)^2}\mathrm{d}\tau.$$

The theorem is proved. ■

Theorem 1.4.6 *Let D be a complex domain and f a holomorphic function, f : d $\to C$. Consider Γ a simple, closed and smooth curve that bounds the domain Δ such that $\overline{\Delta} = \Delta \cup \Gamma \subset D$. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\tau)}{(\tau-z)^{n+1}}\mathrm{d}\tau,\ \forall z \in \Delta.$$

Proof Based on the Cauchy's formula we have

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)}{\tau - z}\mathrm{d}\tau,\ \forall z \in \Delta.$$

According to the previous theorem, we have

$$f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)}{(\tau-z)^2}\mathrm{d}\tau,\ \forall z \in \Delta.$$

Since f' is a monogeneous function we obtain

$$f''(z) = \frac{2!}{2\pi i}\int\limits_{\Gamma}\frac{f(\tau)}{(\tau - z)^3}\mathrm{d}\tau,\ \forall z \in \Delta.$$

The general result can be obtained by using the mathematical induction. The proof of the theorem is closed. ■

1.5 Complex Series

Definition 1.5.1 An expression of the form

$$\sum_{n\geq 0} c_n(z - z_0)^n = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

where the numbers c_n are complex constants, is called a complex series (more explicitely, power complex series), the numbers c_n being referred to as its coefficients.

In the particular case when $z_0 = 0$ we have the power series centred in the origin.

As in the real case, we define the set of convergences A, the radius of convergences ϱ and the disc of convergences as

$$A = \left\{ z \in C : \sum_{n\geq 0} c_n z^n \text{ is convergent} \right\},$$

$$\varrho = \sup_{|z|\in \overline{R}} A,$$

$$\Delta(0, \varrho) = \{z \in C : |z| < \varrho\}.$$

By using the same procedure as in the case of the real power series, we can prove the following theorem.

Theorem 1.5.1 *Denoting by l the following limit*

$$l = \lim_{n\to\infty} \sqrt[n]{|c_n|},$$

then

(1) if $l \in (0, \infty) \Rightarrow \varrho = 1/l$;
(2) if $l = 0 \Rightarrow \varrho = \infty$;
(3) if $l = \infty \Rightarrow \varrho = 0$.

Definition 1.5.2 Let D be a complex domain, f a holomorphic function, $f : \mathrm{d} \to C$ and $z_0 \in D$. A power series of the form

$$\sum_{n\geq 0} c_n(z-z_0)^n, \text{ where } c_n = \frac{1}{n!}f^{(n)}(z_0),$$

is called the Taylor's series relative to point z_0.

Theorem 1.5.2 *Let D, f, z_0 be as in above definitions. Consider the disc having z_0 as center and radius a, $\Delta = \{z \in D : |z-z_0| < a\}$, with boundary Γ. If $\overline{\Delta} = \Delta \cup \Gamma \subset D$ then the Taylor's series attached to the function f relative to the point z_0 is convergent and we have*

$$f(z) = \sum_{n\geq 0} c_n(z-z_0)^n, \text{ where } c_n = \frac{1}{n!}f^{(n)}(z_0).$$

Proof As we already know, since f is holomorphic, it admits derivatives of any order and we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\tau)}{(\tau-z)^{n+1}}\mathrm{d}\tau.$$

Also, by using the Cauchy's formula, we have

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)}{\tau-z}\mathrm{d}\tau. \tag{1.5.1}$$

We can write

$$\frac{1}{\tau-z} = \frac{1}{\tau-z_0-(z-z_0)} = \frac{1}{\tau-z_0}\frac{1}{1-q},$$

where

$$q = \frac{z-z_0}{\tau-z_0}.$$

Because

$$|q| = \left|\frac{z-z_0}{\tau-z_0}\right| = \frac{|z-z_0|}{|\tau-z_0|} = \frac{|z-z_0|}{a} < 1,$$

we can write

$$\frac{1}{1-q} = \frac{1-q^{n+1}+q^{n+1}}{1-q} = \frac{1-q^{n+1}}{1-q} + \frac{q^{n+1}}{1-q} = \sum_{k=0}^{n} q^k + \frac{1}{1-q}q^{n+1}.$$

Therefore

$$\frac{1}{\tau-z} = \frac{1}{\tau-z_0}\left[\sum_{k=0}^{n}\left(\frac{z-z_0)}{\tau-z_0}\right)^k + \frac{1}{1-\frac{z-z_0)}{\tau-z_0}}\left(\frac{z-z_0)}{\tau-z_0}\right)^{n+1}\right] =$$

$$= \sum_{k=0}^{n} \frac{(z - z_0)^k}{(\tau - z_0)^{k+1}} + \frac{1}{\tau - z} \left(\frac{z - z_0)}{\tau - z_0} \right)^{n+1}.$$

By introducing these estimations in Eq. (1.4.1) we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f(\tau) \left[\sum_{k=0}^{n} \frac{(z - z_0)^k}{(\tau - z_0)^{k+1}} + \frac{1}{\tau - z} \left(\frac{z - z_0)}{\tau - z_0} \right)^{n+1} \right] d\tau =$$

$$= \sum_{k=0}^{n} (z - z_0)^k \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\tau)}{(\tau - z_0)^{k+1}} d\tau + R_n(z),$$

where

$$R_n(z) = \frac{1}{2\pi i} (z - z_0)^{n+1} \int_{\Gamma} \frac{f(\tau)}{(\tau - z)(\tau - z_0)^{n+1}} d\tau.$$

Passing to the limit for $n \to \infty$, we obtain $R_n(z) \to 0$ and $f(z)$ becomes

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{n!} f^{(n)}(z_0).$$

The theorem is concluded. ■

Applications 1. Consider the function $f(z) = e^z$ and an arbitrary fixed point $z_0 \in C$. Since $f^{(n)}(z_0) = e^{z_0}, \ \forall n \in N$, we have

$$e^z = \sum_{n=0}^{\infty} e^{z_0} \frac{1}{n!} (z - z_0)^n = e^{z_0} \left(1 + \frac{z - z_0}{1!} + \frac{(z - z_0)^2}{2!} + \cdots + \frac{(z - z_0)^n}{n!} + \cdots \right).$$

In the particular case when $z_0 = 0$ we have

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

2. We want to expand the function $f(z) = \sin z$ around the origin. It is easy to prove that

$$f^n(z) = \sin \left(z + n\frac{\pi}{2} \right) \Rightarrow f^n(0) = \sin \left(n\frac{\pi}{2} \right) = \begin{cases} (-1)^k, \ \text{for } n = 2k + 1 \\ 0, \ \text{for } n = 2k. \end{cases}$$

Then the Taylor's series of the function $f(z) = \sin z$ is

$$\sin z = \sum_{n=0}^{\infty} z^{2n+1} \frac{(-1)^n}{(2n+1)!} = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + +(-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots$$

By using a similar procedure we obtain

$$\cos z = \sum_{n=0}^{\infty} z^{2n} \frac{(-1)^n}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + +(-1)^n \frac{z^{2n}}{(2n)!} + \cdots$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n + \cdots$$

Definition 1.5.3 If the function f is holomorphic on the whole complex plane C then f is called an integer function.

Remarks.

(1) It is easy to prove that the coefficients of a Taylor's series satisfy the inequality

$$|c_n| \le \frac{M(a)}{a^n}, \ \forall n \in N, \ \text{where } M(a) = \sup_{\tau \in \Gamma} |f(\tau)|, \ \Gamma = \{z \in D : |z - \tau| = a\}.$$

(2) Based on the above inequality, Liouville proved that an integer function which is bounded, reduces to a constant.

Definition 1.5.4 A series of the form

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n ,$$

is called a Laurent's series, centred in the point z_0.

We call the main part of a Laurent's series the following series

$$\sum_{n=-\infty}^{-1} c_n (z - z_0)^n ,$$

and the regular part (or Taylor's part) of a Laurent's series the following series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n .$$

Theorem 1.5.3 *Consider the complex domain D, the function f which is holomorphic on D and the corona*

$$\Delta = \{z \in D : R_1 < |z - z_0| < R_2\},$$

having the circles Γ_1 and Γ_2 as boundaries.

If $\overline{\Delta} = \Delta \cup \Gamma_1 \cup \Gamma_2 \subset D$, then the function f admits an expansion in a Laurent's series around z_0

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \; c_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\tau)}{(\tau - z_0)^{n+1}} \mathrm{d}\tau,$$

where Γ is a circle having z_0 as the center and the radius $R \in [R_1, R_2]$.

Proof By using the Cauchy's formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1^- \cup \Gamma_2} \frac{f(\tau)}{\tau - z_0} \mathrm{d}\tau = \frac{1}{2\pi i} \left(\int_{\Gamma_2} \frac{f(\tau)}{\tau - z_0} \mathrm{d}\tau - \int_{\Gamma_1} \frac{f(\tau)}{\tau - z_0} \mathrm{d}\tau \right), \; \forall z \in \Delta. \tag{1.5.2}$$

Consider, separately, the ratio $1/(\tau - z)$:

$$\frac{1}{\tau - z} = \frac{1}{\tau - z_0 - (z_{-}z_0)} = \frac{1}{\tau - z_0} \frac{1}{1 - q},$$

where

$$q = \frac{z - z_0}{\tau - z_0} \Rightarrow |q| = \frac{|z - z_0|}{|\tau - z_0|} = \frac{|z - z_0|}{R_2} < 1.$$

Then

$$\frac{1}{\tau - z} = \frac{1}{\tau - z_0} \left[\sum_{k=0}^{n} \left(\frac{z - z_0}{\tau - z_0} \right)^k + \frac{1}{1 - \frac{z - z_0}{\tau - z_0}} \left(\frac{z - z_0}{\tau - z_0} \right)^{n+1} \right] =$$

$$= \sum_{k=0}^{n} \frac{(z - z_0)^k}{(\tau - z_0)^{k+1}} + \frac{1}{\tau - z} \left(\frac{z - z_0}{\tau - z_0} \right)^{n+1}.$$

Therefore

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\tau)}{\tau - z} \mathrm{d}\tau = \sum_{k=0}^{n} \frac{1}{2\pi i} (z - z_0)^k \int_{\Gamma_2} \frac{f(\tau)}{(\tau - z_0)^{k+1}} \mathrm{d}z + R_n(z),$$

where

$$R_n(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\tau)}{\tau - z} \left(\frac{z - z_0}{\tau - z_0} \right)^{n+1} \mathrm{d}\tau.$$

Passing to the limit by $n \to \infty$ it results $R_n(z) \to 0$. Hence

$$\frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\tau)}{\tau - z} \mathrm{d}\tau = \sum_{k=0}^{\infty} (z - z_0)^k c_k,$$

where

$$c_k = \frac{1}{k!} f^{(k)}(z_0).$$

Let us consider the last term in Eq. (1.5.1)

$$-\frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau)}{\tau - z} \mathrm{d}\tau.$$

Consider, separately, the ratio $-1/(\tau - z)$:

$$-\frac{1}{\tau - z} = \frac{1}{z - \tau} = \frac{1}{z - z_0 - (\tau - z_0)} = \frac{1}{z - z_0} \frac{1}{1 - q},$$

where

$$q = \frac{\tau - z_0}{z - z_0} \Rightarrow |q| = \frac{|\tau - z_0|}{|z - z_0|} = \frac{R_1}{z - z_0} < 1.$$

Then

$$-\frac{1}{\tau - z} = \frac{1}{z - z_0} \left[\sum_{k=1}^{n} \left(\frac{\tau - z_0}{z - z_0} \right)^{k-1} + \left(\frac{\tau - z_0}{z - z_0} \right)^{n} \right] =$$

$$= \sum_{k=1}^{n} \frac{(\tau - z_0)^{k-1}}{(z - z_0)^k} + \frac{1}{z - z_0} \left(\frac{\tau - z_0}{z - z_0} \right)^{n}.$$

Therefore

$$-\frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau)}{\tau - z} \mathrm{d}\tau = \sum_{k=1}^{n} \frac{1}{2\pi i} \frac{1}{(z - z_0)^k} \int\limits_{\Gamma_1} f(\tau)(\tau - z_0)^{k-1} \mathrm{d}z + R_n(z),$$

where

$$R_n(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\tau)}{z - \tau} \left(\frac{\tau - z_0}{z - z_0} \right)^{n} \mathrm{d}\tau.$$

Passing to the limit by $n \to \infty$ it results $R_n(z) \to 0$. Hence

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau = \sum_{k=1}^{\infty}\frac{1}{(z-z_0)^k}\frac{1}{2\pi i}\int\limits_{\Gamma_1} f(\tau)(\tau - z_0)^{k-1}\mathrm{d}z.$$

So, we write

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\tau)}{\tau - z}\mathrm{d}\tau = \sum_{k=-1}^{-\infty}(z-z_0)^k c_k,$$

where

$$c_k = \frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\tau)}{(\tau - z_0)^{k+1}}\mathrm{d}\tau.$$

The proof of the theorem is closed. ∎

Applications 1. Let us find the Laurent's series of the function $e^{1/z}$ around the point $z_0 = 0$. As we know

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

such that, by substituting x by $1/z$ we find

$$e^{\frac{1}{z}} = 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots + \frac{1}{n!}\frac{1}{z^n} + \cdots$$

This series has an infinite number of terms in its main part and only one term in its regular part.

2. Consider the function

$$f(z) = \frac{z}{(z-2)(z+1)^3},$$

and intend to solve the following two problems:

(i) The Laurent's series of the function around the point $z_0 = -1$.
(ii) The Laurent's series of the function in the corona $|z+1| > 3$.

(i). We can write the function in the form

$$f(z) = \frac{1}{(z+1)^3}g(z),$$

where

$$g(z) = \frac{z}{z-2} = \frac{z-2+2}{z-2} = 1 + \frac{2}{z-2} = 1 + \frac{2}{z+1-3} = 1 + \frac{2}{3(\frac{z+1}{3}-1)} =$$

$$= 1 - \frac{2}{3}\frac{1}{1-\frac{z+1}{3}} = 1 - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^n + \cdots\right].$$

Therefore

$$f(z) = \frac{1}{(z+1)^3}\left[1 - \frac{2}{3}\left(1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \cdots + \frac{(z+1)^n}{3^n} + \cdots\right)\right] =$$

$$= \frac{1}{3}\frac{1}{(z+1)^3} - \frac{2}{3^2}\frac{1}{(z+1)^2} - \frac{2}{3^3}\frac{1}{z+1} - \frac{2}{3^4} - \frac{2}{3^5}(z+1) - \frac{2}{3^6}(z+1)^2 - \cdots$$

It is easy to see that the series has an infinite number of terms in its regular part and a finite number (three) in its main part.
(ii). We can write the function in the form

$$f(z) = \frac{1}{(z+1)^3}g(z),$$

where

$$g(z) = \frac{z}{z-2} = \frac{z-2+2}{z-2} = 1 + \frac{2}{z+1-3} == 1 + \frac{2}{(z+1)(1-\frac{3}{z+1})} =$$

$$= 1 + \frac{2}{z+1}\frac{1}{1-\frac{3}{z+1}} = 1 + \frac{2}{z+1}\left[1 + \frac{3}{z+1} + \left(\frac{3}{z+1}\right)^2 + \left(\frac{3}{z+1}\right)^n + \cdots\right].$$

Therefore

$$f(z) = \frac{1}{(z+1)^3} + \frac{2}{(z+1)^4} + \frac{2.3}{(z+1)^5} + \frac{2.3^3}{(z+1)^6} + \cdots$$

It is easy to see that the series has an infinite number of terms in its main part and no one term in its regular part.

Definition 1.5.5 If a complex function has a point that is a singular point and this is not a pole then it is called an essential singular point.

Theorem 1.5.4 *The point z_0 is a pole of order p for the complex function f if and only if the Laurent's series of f around z_0 has the form*

$$f(z) = \frac{c_{-p}}{(z-z_0)^p} + \frac{c_{-p+1}}{(z-z_0)^{p-1}} + \cdots + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots \tag{1.5.3}$$

where $c_p \neq 0$.

Proof Sufficiency. If we suppose that the Laurent's series of f around z_0 has the form (2.3), then we must prove that z_0 is a pole of order p. Hence, we must find a holomorphic complex function φ, $\varphi(z_0) \neq 0$ such that

$$f(z) = \frac{1}{(z-z_0)^p}\varphi(z).$$

To this, we define the function φ by

$$\varphi(z) = c_p + c_{-p+1}(z-z_0) + c_{-p+2}(z-z_0)^2 + \cdots + c_0(z-z_0)^p + c_1(z-z_0)^{p+1}.$$

Of course, being a polynomial, φ is a holomorphic function. Also, it is easy to see that $\varphi(z_0) = c_p \neq 0$. By direct calculations we obtain that

$$\frac{1}{(z-z_0)^p}\varphi(z) = f(z).$$

Necessity. Suppose that z_0 is a pole of order p. Then, there exists a holomorphic function φ such that $\varphi(z_0) \neq 0$ and

$$f(z) = \frac{1}{(z-z_0)^p}\varphi(z).$$

Being a holomorphic function, φ can be expanded in a Taylor's series around z_0:

$$\varphi(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots,$$

where $\varphi(z_0) = a_0 \neq 0$. Multiplying both sides of the above equality by

$$1/(z-z_0)^p$$

we obtain

$$f(z) = \frac{a_0}{(z-z_0)^p} + \frac{a_1}{(z-z_0)^{p-1}} + \cdots + a_p + a_{p+1}(z-z_0) + a_{p+2} + (z-z_0)^2 + \cdots$$

and the theorem is concluded. ∎

Remark. By using this theorem we can say that a singular point is a pole for a function f if the Laurent's series of f has a finite number of terms in its main part.

Of course, if the Laurent's series of f, around z_0, has an infinite number of terms in its main part, we deduce that z_0 is an essential singular point of f. For instance, in a

previous application, we proved that the Laurent'a series of the function $f(z) = e^{1/z}$, around $z_0 = 0$, has an infinite number of terms, such that $z_0 = 0$ is an essential singular point of f.

Definition 1.5.6 If all singular points of a function f are poles then the function f is called a meromorphic function.

By using the definition of a meromorphic function, the readers can prove the properties included in the following proposition.

Proposition 1.5.1 *The following assertions are still valid:*

(1) A meromorphic function on a bounded domain has only a finite number of poles.
(2) The sum, product and quotient of two meromorphic functions are also meromorphic functions.

The following two theorems prove two results regarding an integer function and the point $z_0 = \infty$.

Theorem 1.5.5 *An integer function f has the point $z_0 = \infty$ as an essential singular point if and only if f is not a polynomial.*

Proof We write the Taylor's series around the origin $z_0 = 0$, for $|z| < R$, with a high enough R:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$$

Define the function φ by

$$\varphi(\tau) = f(\frac{1}{\tau} = c_0 + \frac{c_1}{\tau} + \frac{c_2}{\tau^2} + \cdots + \frac{c_n}{\tau^n} + \cdots$$

Then the point $z_0 = \infty$ is an essential singular point of the function f if the point $\tau = 0$ is an essential singular point of the function φ and that is possible if and only if the Laurent's series of φ has in its main part an infinite number of terms, i.e. f is not a polynomial and the theorem is concluded. ■

Theorem 1.5.6 *If an integer function f has the point $z_0 = \infty$ as an ordinary point, then the function f is a constant.*

Proof If the function f has the point $z_0 = \infty$ as an ordinary point, then the function $\varphi(\tau) = f(1/\tau)$ has the point $\tau_0 = 0$ as an ordinary point. Then the main part of the Laurent's series of φ is null. But

$$\varphi(\tau) = c_0 + \frac{c_1}{\tau} + \frac{c_2}{\tau^2} + \cdots + \frac{c_1}{\tau} + \cdots$$

Then

$$c_n = 0, \ \forall n > 1 \Rightarrow \varphi(\tau) = c_0 \Rightarrow f(z) = c_0.$$

The theorem is proved. ■

The following result is called *the Fundamental Theorem of Algebra* and is due to D′Alembert and Gauss.

Theorem 1.5.7 *(D′Alembert-Gauss) Any polynomial of order* $n \geq 1$ *has at least one complex root.*

Proof Assuming, by contrary, that $P(z) \neq 0, \ \forall z \in C$ then the function

$$f(z) = \frac{1}{P(z)}$$

is an integer function which has the point $z_0 = \infty$ as an ordinary point.

Therefore, f reduces to a constant and, as a consequence, the polynomial P reduces to a constant, that contradicts the assumption that the order of P is $n \geq 1$. The theorem is concluded. ■

Definition 1.5.7 Let D be a complex domain, f a holomorphic function $f : D \to C$ and $a \in C$ a pole or an essential singular point of f. Consider the disc $\Delta(a, R)$ having as boundary the circle $\Gamma(a, R)$, with the radius R such that $\overline{\Delta}\backslash\{a\} \subset D$, where $\overline{\Delta} = \Delta \cup \Gamma$.
Then, the value of the integral

$$\frac{1}{2\pi i} \int_{\Gamma} f(z) \mathrm{d}z$$

is called the residue of the function f at the point a and is denoted by

$$\mathrm{res}(f, a) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \mathrm{d}z.$$

Theorem 1.5.8 *Let* D *be a complex domain,* f *a holomorphic function* $f : D \to C$ *and* $a \in C$ *an isolated singular point of* f*. To compute the residue of the function* f *at* a *we have the following three possibilities:*

(1) If we write the Laurent's series of f *around* a

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

then $\mathrm{res}(f, a) = c_1$.

(2) If a *is a pole of order* p *of the function* f*, then*

$$\mathrm{res}(f, a) = \frac{1}{(p-1)!}\varphi^{(p-1)}(a), \ \varphi(z) = \begin{cases} (z-a)^p f(z), \ \text{if } z \in D\backslash\{a\} \\ \lim\limits_{z \to a}(z-a)^p f(z), \ \text{if } z = a. \end{cases}$$

(3) *If a is a pole or order 1 (called a simple pole) of the function f that is a quotient of two functions $f = g/h$. With other words, a is an ordinary point for g and h and, in addition, $h(a) \neq 0$. Then, the residue can be calculated by the formula*

$$\operatorname{res}(f, a) = \frac{g(a)}{h'(a)}.$$

Proof 1. Clearly, the formula proposed at point 1. is valid both in the case when a is a pole and in the case when a is an essential singular point. We write the Laurent's series of f around the point a

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \ c_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z.$$

Based on the Definition 1.5.7, we obtain

$$\operatorname{res}(f, a) = c_1 = \frac{1}{2\pi i}\int_\Gamma f(z)\mathrm{d}z.$$

2. If a is a pole of order p, then there esists the holomorphic function φ with $\varphi(a) \neq 0$ such that

$$f(z) = \frac{1}{(z-a)^p}\varphi(z) \Rightarrow \frac{1}{2\pi i}\int_\Gamma f(z)\mathrm{d}z = \frac{1}{2\pi i}\int_\Gamma \frac{1}{(z-a)^p}\varphi(z)\mathrm{d}z.$$

Now, we use the Cauchy's formula for derivative

$$\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(z)}{(z-a)^p}\mathrm{d}z = \frac{1}{(p-1)!}\frac{(p-1)!}{2\pi i}\int_\Gamma \frac{\varphi(z)}{(z-a)^p}\mathrm{d}z = \frac{1}{(p-1)!}\varphi^{(p-1)}(a).$$

3. Since g and h are holomorphic functions we can write the Taylor'a series around a such that

$$f(z) = \frac{c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots}{d_0 + d_1(z-a) + d_2(z-a)^2 + \cdots}$$

Because a is a pole for f we have $h(a) = 0$ and then $d_0 = 0$. Thus

$$f(z) = \frac{c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots}{d_1(z-a) + d_2(z-a)^2 + \cdots} \Rightarrow$$

$$\Rightarrow \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\frac{c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots}{d_1 + d_2(z-a) + d_3(z-a)^2 + \cdots} = \frac{c_0}{d_1} = \frac{g(a)}{h'(a)}.$$

The proof of the theorem is closed. ■

Applications 1. Let us compute the residues at points $a = 2$ and $a = -1$ for the function

$$f(z) = \frac{z}{(z-2)(z+1)^3}.$$

Since $a = 2$ is a simple pole, we have

$$\text{res}(f, 2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{z}{(z+1)^3} = \frac{2}{27}.$$

Since $a = -1$ is a pole of order three, we have

$$\text{res}(f, -1) = \lim_{z\to -1}\frac{1}{2!}\left((z-1)^3 f(z)\right)'' = \lim_{z\to -1}\frac{1}{2!}\left(\frac{z}{z-2}\right)'' = -\frac{2}{27}$$

2. Let us compute the residue at point $a = 0$ for the function

$$f(z) = z^k e^{\frac{1}{z}}$$

Since $a = 0$ is an essential singular point for the given function, we must write its Laurent's series as follows:

$$e^{\frac{1}{z}} = 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots + \frac{1}{k!}\frac{1}{z^k} + \frac{1}{(k+1)!}\frac{1}{z^{k+1}} + \cdots$$

$$f(z) = z^k + \frac{1}{1!}z^{k-1} + \frac{1}{2!}z^{k-2} + \cdots + \frac{1}{k!} + \frac{1}{(k+1)!}\frac{1}{z} + \cdots$$

Then for the residue we obtain

$$\text{res}(f, 0) = c_1 = \frac{1}{(k+1)!}.$$

The following result, called *The Theorem of Residues*, is a fundamental result in the theory of complex functions.

Theorem 1.5.9 *Let Δ be a bounded domain with boundary Γ which is a simple, closed and smooth curve. Consider a function f that has a finite number of singular points (poles or essential singular points) $S = \{a_1, a_2, \ldots, a_n\}$. If the function f is holomorphic on a domain D such that $\overline{\Delta}\backslash S \subset D$, $\overline{\Delta} = \Delta \cup \Gamma$, then*

$$\int_\Gamma f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \text{res}(f, a_k).$$

Proof Around the points a_k, consider the discs δ_k having as boundaries the circles γ_k such that $\delta_k \cup \gamma_k = \overline{\delta}_k$ are disjunctive. Since f is a holomorphic function on

$$\overline{\Delta}\setminus\bigcup_{k=1}^{n}\delta_k$$

we have

$$\int_{\Gamma} f(z)\mathrm{d}z + \sum_{k=1}^{n}\int_{\gamma_k^-} f(z)\mathrm{d}z = 0 \Rightarrow \int_{\Gamma} f(z)\mathrm{d}z = \sum_{k=1}^{n}\int_{\gamma_k} f(z)\mathrm{d}z.$$

But, by definition

$$\mathrm{res}(f, a_k) = \frac{1}{2\pi i}\int_{\Gamma} f(z)\mathrm{d}z.$$

Therefore

$$\int_{\Gamma} f(z)\mathrm{d}z = 2\pi i\sum_{k=1}^{n}\mathrm{res}(f, a_k)$$

and the theorem is concluded. ■

It is possible that a function f has a large number of singular points and then we must compute a large number of residues. To avoid this trouble we introduce the residue of f at $z_0 = \infty$.

Definition 1.5.8 Let f be a monogeneous function in all points outside to the disc $\Delta(0, R_0)$ such that the point $z_0 = \infty$ is an ordinary point of f, or an isolated pole or essential singular point. The number denoted by $res(f, \infty)$ and defined by

$$\mathrm{res}(f, \infty) = -\frac{1}{2\pi i}\int_{\Gamma} f(z)\mathrm{d}z$$

is called the residue of f to infinity. Here Γ is a circle having the origin as center and the radius R such that $R > R_0$.

If we write the Laurent's series of f in the corona $R_0 < |z| < R_1$, where R_1 is sufficient large

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n,\ c_n = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z^{n+1}}\mathrm{d}z,$$

then the residue of f to infinity is $\mathrm{res}(f, \infty) = -c_1$.

Theorem 1.5.10 *Let E be a complex set and $f : E \to \overline{C}$ that has a finite number of singular points. Then the sum of all residues of f is zero:*

$$\operatorname{res}(f, \infty) + \sum_{k=1}^{n} \operatorname{res}(f, a_k) = 0$$

where a_k are the singular points of f.

Proof Consider the disc Δ having the origin as center and the radius R_0, sufficient large such that Δ includes all the singular points of f. If the circle Γ is the boundary of Δ, by using the residues theorem, we have

$$\frac{1}{2\pi i} \int_{\Gamma} f(z)\mathrm{d}z = \sum_{k=1}^{n} \operatorname{res}(f, a_k).$$

But

$$-\frac{1}{2\pi i} \int_{\Gamma} f(z)\mathrm{d}z = \operatorname{res}(f, \infty).$$

Therefore

$$-\operatorname{res}(f, \infty) = \sum_{k=1}^{n} \operatorname{res}(f, a_k) \Rightarrow \operatorname{res}(f, \infty) + \sum_{k=1}^{n} \operatorname{res}(f, a_k) = 0.$$

The theorem is concluded. ■

In the following we will use some procedure to compute some real improper integral with the aid of the residues.

Firstly, we prove two auxiliary results, included in the following propositions due to Jordan.

Proposition 1.5.2 (Jordan) *Let $\widehat{AB}$ be an arc of the circle $|z| = R$ such that $\alpha \leq \arg z \leq \beta$. If*

$$\lim_{|z|\to\infty} zf(z) = k, \ \ k = \text{constant},$$

then

$$\lim_{|z|\to\infty} \int_{AB} f(z)\mathrm{d}z = i(\beta - \alpha)k.$$

Proof We can write $zf(z) = k + \varphi(z)$, where φ has the property that $\forall \varepsilon > 0$ we have $|\varphi(z)| < \varepsilon$, for $|z| \to \infty$, i.e. $\varphi(z) \to 0$, for $|z| \to \infty$. If we write

$$f(z) = \frac{k}{z} + \frac{\varphi(z)}{z},$$

then

$$\int_{AB} f(z)\mathrm{d}z = \int_{AB} \frac{k}{z}\mathrm{d}z + \int_{AB} \frac{\varphi(z)}{z}\mathrm{d}z.$$

By using the polar coordinates, we obtain

$$\int_{AB} f(z)\mathrm{d}z = \int_{\alpha}^{\beta} \frac{k}{Re^{i\theta}} iRe^{i\theta}\mathrm{d}\theta + \int_{\alpha}^{\beta} \frac{\varphi(Re^{i\theta})}{Re^{i\theta}} iRe^{i\theta}\mathrm{d}\theta =$$

$$= ik(\beta - \alpha) + i\int_{\alpha}^{\beta} \varphi(Re^{i\theta})\mathrm{d}\theta.$$

Therefore

$$\left|\int_{AB} f(z)\mathrm{d}z - ik(\beta - \alpha)\right| = \left|\int_{\alpha}^{\beta} \varphi(Re^{i\theta})\mathrm{d}\theta\right| \leq \int_{\alpha}^{\beta} \varphi(Re^{i\theta})\mathrm{d}\theta < \varepsilon(\beta - \alpha),$$

and the proposition is concluded. ■

Proposition 1.5.3 (Jordan) *Let* $\widehat{AB}$ *be an arc of the circle* $|z - a| = r$ *such that* $\alpha \leq arg z \leq \beta$. *If*

$$\lim_{|z|\to a} (z - a) f(z) = k, \ \ k = \text{constant},$$

then

$$\lim_{|z|\to a} \int_{AB} f(z)\mathrm{d}z = i(\beta - \alpha)k.$$

Proof We can write $zf(z) = k + \varphi(z)$, where φ has the property that $\forall \varepsilon > 0$ we have $|\varphi(z)| < \varepsilon$, for $|z| \to a$, i.e. $\varphi(z) \to 0$, for $|z| \to a$. If we write

$$f(z) = \frac{k}{z - a} + \frac{\varphi(z)}{z - a},$$

then

$$\int_{AB} f(z)\mathrm{d}z = \int_{AB} \frac{k}{z - a}\mathrm{d}z + \int_{AB} \frac{\varphi(z)}{z - a}\mathrm{d}z.$$

By using the polar coordinates, $z = a + re^{i\theta}$, we obtain

$$\int\limits_{AB} f(z)\mathrm{d}z = \int\limits_{\alpha}^{\beta} \frac{k}{re^{i\theta}} ire^{i\theta}\mathrm{d}\theta + \int\limits_{\alpha}^{\beta} \frac{\varphi(r+ae^{i\theta})}{re^{i\theta}} ire^{i\theta}\mathrm{d}\theta =$$

$$= ik(\beta-\alpha) + i\int\limits_{\alpha}^{\beta} \varphi(a+re^{i\theta})\mathrm{d}\theta.$$

Therefore

$$\left|\int\limits_{AB} f(z)\mathrm{d}z - ik(\beta-\alpha)\right| = \left|\int\limits_{\alpha}^{\beta} \varphi(a+re^{i\theta})\mathrm{d}\theta\right| \le \int\limits_{\alpha}^{\beta} \varphi(a+re^{i\theta})\mathrm{d}\theta < \varepsilon(\beta-\alpha).$$

The proposition is concluded. ■

In the last part of this chapter we indicate several real improper integrals which can be calculate by using the previous two propositions and the Theorem of Residues.

I. Let us consider an integral of the form

$$\int\limits_{\alpha}^{\alpha+2\pi} R(\cos\theta, \sin\theta)\mathrm{d}\theta.$$

By using the substitution

$$z = e^{i\theta},\ \ \theta \in [\alpha, \alpha+2\pi]$$

we deduce that z lies on the circle Γ having the origin as the center. Also,

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right),\ \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right),\ \mathrm{d}z = ie^{i\theta}\mathrm{d}\theta \Rightarrow \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}.$$

The given integral becomes

$$\int\limits_{\Gamma} \frac{1}{iz} R\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z+\frac{1}{z}\right)\right]\mathrm{d}z = 2\pi i\sum_{k} \mathrm{res}(R_1, a_k),$$

where the function R_1 is

$$R_1(z) = \frac{1}{iz}R(z)$$

and a_k are the singular points of this function.

Application. Let us compute the integral

$$I_1 = \int_{-\pi}^{\pi} \frac{1 + 2\cos\theta}{5 + 4\sin\theta} d\theta.$$

With the above procedure, we have

$$I_1 = \int_{\gamma} \frac{1}{iz} \frac{1 + z + 1/z}{5 + (z - 1/z)2/i} = \int_{\gamma} \frac{\left(z^2 + z + 1\right) iz}{iz^2(5iz + 2z^2 - 2)} dz =$$

$$= \int_{\gamma} \frac{z^2 + z + 1}{z(2z^2 + 5iz - 2)} dz = 2\pi i \left[\mathrm{res}(f, 0) + \mathrm{res}\left(f, -\frac{i}{2}\right)\right].$$

II. Let us consider an integral of the form

$$I_2 = \int_{-\infty}^{\infty} R(x)dx,\ \ R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials such that $Q(x) \neq 0, \forall x \in R$ and 1+degree(P) <degree(Q). In order to find the value of I_2 we write

$$I_2 = \lim_{R\to\infty} \int_{-R}^{R} R(x)dx.$$

We use the domain bounded by the curve determined by the superior semi-circle Γ having the origin as the center and the radius equal to R together with the segment $[-R, R]$ and integrate on this curve the function $f(z) = R(z)$:

$$\int_{\Gamma} R(z)dz + \int_{-R}^{R} R(x)dx = 2\pi i \sum_k \mathrm{res}(f, a_k), \tag{1.5.4}$$

where a_k are the singular points of f which lie in the superior half plane $y = Im(z) > 0$.

By using the hypotheses and the first Jordan's results, we obtain

$$zf(z) = \frac{zP(z)}{Q(z)} \Rightarrow \lim_{R\to\infty} zf(z) = \lim_{R\to\infty} \frac{zP(z)}{Q(z)} = 0.$$

Passing to the limit in Eq. (1.5.3) for $R \to \infty$ we obtain $\int\limits_{\Gamma} R(z)\mathrm{d}z \to 0$ such that Eq. (1.5.3) reduces to

$$I_2 = \int\limits_{-\infty}^{\infty} R(x)\mathrm{d}x = 2\pi i \sum_k \mathrm{res}(f, a_k).$$

Application. Let us compute the integral

$$I_2 = \int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^4}\mathrm{d}x.$$

By using the above result, we deduce that

$$\int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^4}\mathrm{d}x = 2\pi i \left[\mathrm{res}(f, z_1) + \mathrm{res}(f, z_2)\right],$$

where z_1 and z_2 are the roots of the equation

$$z^4 + 1 = 0$$

having $Im(z) > 0$, i.e.

$$z_1 = e^{i\pi/4},\ z_2 = e^{i(\pi/4+\pi/2)}.$$

III. Let us consider an integral of the form

$$I_3 = \int\limits_0^{\infty} x^{\alpha} R(x)\mathrm{d}x,\ \ \alpha \in (-1) \cup (0, 1),\ \ R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials such that $Q(x) \neq 0$, $\forall x \in [0, \infty)$ and $1 + \alpha +$degree$(P) <$degree(Q).

In order to find the value of I_3 we write

$$I_2 = \lim_{r \to 0 R \to \infty} \int\limits_r^R x^{\alpha} R(x)\mathrm{d}x.$$

We use the corona bounded by the circle Γ having the origin as the center and the radius equal to R and the circle γ having the origin as the center and the radius equal to r. We make a cut in this corona along the x-axis and integrate on this domain the function

$$f(z) = z^\alpha R(z):$$

$$\int_r^R x^\alpha R(x)\mathrm{d}x + \int_\Gamma z^\alpha R(z)\mathrm{d}z + e^{2\pi i\alpha}\int_R^r x^\alpha R(x)\mathrm{d}x + \\ + \int_\gamma z^\alpha R(z)\mathrm{d}z = 2\pi i \sum_k \mathrm{res}(f, a_k), \tag{1.5.5}$$

where a_k are the singular points of f.

By using the hypotheses and the Jordan's results, we obtain

$$\lim_{R\to\infty} zf(z) = \lim_{R\to\infty}\frac{z^{1+\alpha}P(z)}{Q(z)} = 0 \Rightarrow \lim_{R\to\infty}\int_\Gamma z^\alpha R(z)\mathrm{d}z = 0,$$

$$\lim_{r\to 0} zf(z) = \lim_{r\to 0}\frac{z^{1+\alpha}P(z)}{Q(z)} = 0 \Rightarrow \lim_{r\to 0}\int_\gamma z^\alpha R(z)\mathrm{d}z = 0.$$

Taking into account these results, by passing to the limit in Eq. (1.5.4) for $R \to \infty$ and $r \to 0$, Eq. (1.5.4) reduces to

$$\left(1 - e^{2\pi i\alpha}\right)\int_0^\infty x^\alpha R(x)\mathrm{d}x = 2\pi i \sum_k \mathrm{res}(f, a_k).$$

Application. Let us compute the integral

$$I_3 = \int_0^\infty \frac{\sqrt{x}}{1+x^3}\mathrm{d}x.$$

Here $\alpha = 1/2$ and, by using the above result, we deduce that

$$I_3 = \frac{2\pi i}{1 - e^{\pi i}}\left[\mathrm{res}(f, z_1) + \mathrm{res}(f, z_2) + \mathrm{res}(f, z_3)\right],$$

where z_1, z_2 and z_3 are the roots of the equation $z^3 + 1 = 0$.

IV. Let us consider an integral of the form

$$I_4 = \int_0^\infty x^\alpha R(x)\ln(x)\mathrm{d}x,\ \ \alpha \in (-1) \cup (0, 1),\ \ R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials such that $Q(x) \neq 0$, $\forall x \in [0,\infty)$ and $1+\alpha+$degree$(P) <$degree(Q).

In order to find the value of I_4 we write

$$I_4 = \lim_{r\to 0 R\to\infty} \int_r^R x^\alpha R(x)\ln(x)\mathrm{d}x.$$

We use the corona bounded by the circle Γ having the origin as the center and the radius equal to R and the circle γ having the origin as the center and the radius equal to r. We make a cut in this corona along the x-axis and integrate on this domain the function $f(z) = z^\alpha R(z)\ln(z)$:

$$\int_r^R x^\alpha R(x)\ln(x)\mathrm{d}x + \int_\Gamma z^\alpha R(z)\ln(z)\mathrm{d}z + e^{2\pi i\alpha}\int_R^r x^\alpha R(x)[\ln(x)+2\pi i]\mathrm{d}x + \\ + \int_\gamma z^\alpha R(z)\ln(z)\mathrm{d}z = 2\pi i\sum_k \mathrm{res}(f, a_k), \tag{1.5.6}$$

where a_k are the singular points of f.

By using the hypotheses and the Jordan's results, we obtain

$$\lim_{R\to\infty} zf(z) = \lim_{R\to\infty}\frac{z^{1+\alpha}P(z)}{Q(z)}\ln(z) = 0 \Rightarrow \lim_{R\to\infty}\int_\Gamma z^\alpha R(z)\ln(z)\mathrm{d}z = 0,$$

$$\lim_{r\to 0} zf(z) = \lim_{r\to 0}\frac{z^{1+\alpha}P(z)}{Q(z)}\ln(z) = 0 \Rightarrow \lim_{r\to 0}\int_\gamma z^\alpha R(z)\ln(z)\mathrm{d}z = 0.$$

Taking into account these results, by passing to the limit in Eq. (1.5.5) for $R\to\infty$ and $r\to 0$, Eq. (1.5.5) reduces to

$$\left(1-e^{2\pi i\alpha}\right)\int_0^\infty x^\alpha R(x)\ln(x)\mathrm{d}x - 2\pi i\int_0^\infty x^\alpha R(x)\mathrm{d}x = 2\pi i\sum_k \mathrm{res}(f, a_k),$$

where $f(z) = z^\alpha R(z)\ln(z)$.

Application. Let us compute the integral

$$I_4 = \int_0^\infty \frac{\ln x}{\sqrt{x}(x^2+a^2)}\mathrm{d}x,\ a > 0.$$

Here $\alpha = -1/2$ and f is the function

$$f(z) = \frac{\ln z}{\sqrt{z}(z^2 + a^2)}.$$

By using the above theoretical result, we deduce that

$$\left(1 - e^{-\pi i}\right) \int_0^\infty \frac{\ln x}{\sqrt{x}(x^2 + a^2)} \mathrm{d}x - 2\pi i e^{-\pi i} \int_0^\infty \frac{1}{\sqrt{x}(x^2 + a^2)} \mathrm{d}x =$$

$$= 2\pi i \left[\operatorname{res}(f, z_1) + \operatorname{res}(f, z_2) + \operatorname{res}(f, z_3)\right].$$

The integral

$$J = \int_0^\infty \frac{1}{\sqrt{x}(x^2 + a^2)} \mathrm{d}x$$

can be computed by using the procedure of I_3, by using the function

$$g(z) = \frac{1}{\sqrt{z}(z^2 + a^2)}$$

such that we obtain

$$J = \frac{2\pi i}{1 - e^{-\pi i}} \left[\operatorname{res}(f, z_1) + \operatorname{res}(f, z_2) + \operatorname{res}(f, z_3)\right].$$

Chapter 2
Special Functions

2.1 Euler's Functions

Euler's function of first species $\boldsymbol{\Gamma}$

Definition 2.1.1 Consider the semi-plane $\Delta_0 = \{z \in C,\ z = x + iy : x > 0\}$. The complex function $\Gamma : \Delta_0 \to C$ defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t,$$

is called the Euler's function of first species.

Remark. Since the Euler's function of first species is defined as an improper integral, we first must prove that it is well defined. This result and the main properties are included in the following theorem.

Theorem 2.1.1 *The function* Γ *satisfies the following properties:*
(1) Γ *is well defined, i.e.*

$$\left| \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t \right| < \infty;$$

(2) Γ *is a holomorphic function on* Δ_0*;*
(3) $\Gamma(z+1) = z\Gamma(z)$, $\forall z \in \Delta_0$. *As a consequence, we have* $\Gamma(n+1) = n!$, $\forall n \in N$.

Proof (1) We use the well known formula

$$u^v = e^{v \ln u}.$$

Therefore

$$t^{z-1} = e^{(z-1)\ln t} = e^{(x-1)\ln t + iy \ln t} =$$

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$$= e^{(x-1)\ln t}\left[\cos(y \ln t) + i \sin(y \ln t)\right].$$

Then we obtain

$$\left|t^{z-1}\right| = e^{(x-1)\ln t} = t^{x-1}.$$

Hence

$$\left|\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t\right| \le \int_0^\infty \left|t^{z-1}\right|\left|e^{-t}\right|\mathrm{d}t =$$

$$= \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t = \int_0^1 t^{x-1}e^{-t}\mathrm{d}t + \int_1^\infty t^{x-1}e^{-t}\mathrm{d}t = I_1 + I_2.$$

For the integral I_1 we have:

$$0 < t < 1 \Rightarrow 0 > -t > -1 \Rightarrow e^{-t} < 1 \Rightarrow t^{x-1}e^{-t} < t^{x-1} \Rightarrow$$

$$\Rightarrow I_1 \le \int_0^1 t^{x-1}\mathrm{d}t = \left.\frac{t^x}{x}\right|_0^1 = \frac{1}{x} < \infty.$$

We make now some estimations on the integral I_2. As we know

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^m}{m!} + \ldots \Rightarrow$$

$$\Rightarrow e^t \ge \frac{t^m}{m!} \Rightarrow e^{-t} \le \frac{m!}{t^m}.$$

If we choose $m > x$, it results

$$I_2 \le \int_1^\infty t^{x-1}\frac{m!}{t^m}\mathrm{d}t = m!\int_1^\infty t^{x-m-1}\mathrm{d}t =$$

$$= m!\left.\frac{t^{x-m}}{x-m}\right|_1^\infty = \frac{m!}{m-x} < \infty.$$

Finally, we obtain

$$|\Gamma(z)| \le I_1 + I_2 \le \frac{1}{x} + \frac{m!}{m-x} < \infty.$$

(2) We write Γ in the usual form of a complex function $\Gamma(z) = u(x, y) + iv(x, y)$ and verify the Cauchy–Riemann's condition:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

By using the Cauchy's formula for e^z we have

$$\Gamma(z) = \int_0^\infty t^{x-1}e^{-t}\left[\cos(y\ln t) + i\sin(y\ln t)\right]\mathrm{d}t =$$

$$= \int_0^\infty t^{x-1}e^{-t}\cos(y\ln t)\mathrm{d}t + i\int_0^\infty t^{x-1}e^{-t}\sin(y\ln t)\mathrm{d}t = u(x,y) + iv(x,y).$$

Thus

$$\frac{\partial u}{\partial x} = \int_0^\infty t^{x-1}\ln t e^{-t}\cos(y\ln t)\mathrm{d}t,\ \ \frac{\partial v}{\partial y} = \int_0^\infty t^{x-1}\ln t e^{-t}\cos(y\ln t)\mathrm{d}t,$$

such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

In the same manner we can prove the second Cauchy–Riemann's condition.

(3) Substituting z by $z+1$ we obtain

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\mathrm{d}t,$$

such that, integrating by parts, it results

$$\Gamma(z+1) = -t^z e^{-t}\Big|_0^\infty + \int_0^\infty z t^{z-1}e^{-t}\mathrm{d}t = z\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t = z\Gamma(z),$$

since $\lim\limits_{t\to\infty} t^z/e^t = 0$.

In the particular case $z = n \in N$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\Gamma(1).$$

By using the definition, we have

$$\Gamma(1) = \int_0^\infty t^0 e^{-t} \mathrm{d}t = \int_0^\infty e^{-t} \mathrm{d}t = -e^{-t}\Big|_0^\infty = 1$$

and the theorem is concluded. ■

Euler's function of second species β

Definition 2.1.2 Let p, q be two complex numbers such that $Re(p) > 0$ and $Re(q) > 0$, i.e. $p, q \in \Delta_0$ (see the above definition of Δ_0). The function $\beta : \Delta_0 \times \Delta_0 \to C$ defined by

$$\beta(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} \mathrm{d}t,$$

is called the Euler's function of second species.

In the following theorem we prove the main properties of β as well as the connection between β and Γ.

Theorem 2.1.2 *The function Γ satisfies the following properties:*
(1) $\beta(p, q) = \beta(q, p)$;
(2) $p\beta(p, q+1) = q\beta(p+1, q)$;
(3) $\beta(p, q)\Gamma(p+q) = \Gamma(p)\Gamma(q)$.

Proof (1) It is easy to prove this commutative property with the aid of the substitution $1 - t = \tau$.

(2) By direct calculations we obtain

$$p\beta(p, q+1) = p \int_0^1 t^{p-1}(1-t)^q \mathrm{d}t = \int_0^1 \left(t^p\right)' (1-t)^q \mathrm{d}t =$$

$$= t^p (1-t)^q \Big|_0^1 + \int_0^1 q t^p (1-t)^{q-1} \mathrm{d}t = q\beta(p+1, q).$$

(3) We start by using the right-hand side term

$$\Gamma(p)\Gamma(q) = \int_0^\infty y^{p-1} e^{-y} \mathrm{d}y \int_0^\infty x^{q-1} e^{-x} \mathrm{d}x =$$

$$= \int_0^\infty \int_0^\infty y^{p-1} e^{-y} x^{q-1} e^{-x} \mathrm{d}x \mathrm{d}y.$$

Let us change in the last integral the variables as follows:
$x = u^2 \Rightarrow \mathrm{d}x = 2u\mathrm{d}u$. For $x = 0 \Rightarrow u = 0$ and for $x = \infty \Rightarrow u = \infty$
$y = v^2 \Rightarrow \mathrm{d}x = 2v\mathrm{d}v$. For $y = 0 \Rightarrow v = 0$ and for $y = \infty \Rightarrow v = \infty$.
So, the last integral becomes

$$\Gamma(p)\Gamma(q) = \int_0^\infty \left(u^2\right)^{p-1} e^{-u^2} 2u\mathrm{d}u \int_0^\infty \left(v^2\right)^{q-1} e^{-v^2} 2v\mathrm{d}v =$$

$$= 4 \int_0^\infty \int_0^\infty e^{-\left(u^2+v^2\right)} u^{2p-1} v^{2q-1} \mathrm{d}u\mathrm{d}v.$$

Now, we use the polar coordinates

$$\begin{cases} u = \varrho \cos\theta, \ 0 \le \theta \le \frac{\pi}{2} \\ v = \varrho \sin\theta, \ 0 \le \varrho < \infty \end{cases} \Rightarrow \frac{D(u, v)}{D(\varrho, \theta)} = \varrho.$$

Thus, our integral becomes

$$\Gamma(p)\Gamma(q) = 4 \int_0^\infty \int_0^{\pi/2} e^{-\varrho^2} \varrho^{2p-1} (\cos\theta)^{2p-1} \varrho^{2q-1} (\sin\theta)^{2q-1} \varrho \mathrm{d}\theta \mathrm{d}\varrho = 4\left(I_1.I_2\right).$$

By using the new variable a defined by $a = \varrho^2 p \Rightarrow \mathrm{d}a = 2\varrho\mathrm{d}\varrho$, the integral I_1 becomes

$$I_1 = \frac{1}{2} \int_0^\infty e^{-a} a^{p+q-1} \mathrm{d}a = \frac{1}{2}\Gamma(p+q).$$

We use now the new variable b defined by $b = \cos^2\theta \Rightarrow \mathrm{d}b = -2\sin\theta\cos\theta\mathrm{d}\theta$. So, for $\theta = 0 \Rightarrow b = 1$ and for $\theta = \pi/2 \Rightarrow b = 0$. Thus, the integral I_2 becomes

$$I_2 = \int_1^0 \frac{b^p}{\cos\theta} \frac{(1-b)^q}{\sin\theta} \frac{\mathrm{d}b}{-2\cos\theta\sin\theta} = \frac{1}{2} \int_0^1 \frac{b^p (1-b)^q}{\cos^2\theta \sin^2\theta} \mathrm{d}b =$$

$$= \frac{1}{2} \int_0^1 \frac{b^p(1-b)^q}{b(1-b)} \mathrm{d}b = \frac{1}{2} \int_0^1 b^{p-1} (1-b)^{q-1} \mathrm{d}b = \frac{1}{2}\beta(p, q).$$

Multiplying the integrals I_1 and I_1 we obtain

$$\Gamma(p)\Gamma(q) = 4\frac{1}{2}\Gamma(p+q)\frac{1}{2}\beta(p,q) \Rightarrow$$

$$\Rightarrow \beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

The theorem is concluded. ■

Application. Let us compute the integral

$$I = \int_a^b \frac{1}{\sqrt{(b-x)(x-a)}}\mathrm{d}x.$$

Introduce a new variable t by

$$b - x|_a^b = (b-a)t|_1^0 \Rightarrow \mathrm{d}x = (a-b)\mathrm{d}t,\ x - a = (b-a)(1-t).$$

Thus, the integral becomes

$$I = \int_1^0 \frac{a-b}{\sqrt{t(1-t)(b-a)}}\mathrm{d}t = \int_0^1 \frac{1}{\sqrt{t(1-t)}}\mathrm{d}t = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$$

We now prove a very important result that is known as *the complements formula* and that is very useful in many applications.

Theorem 2.1.3 *For $p \notin Z$, the function $\Gamma(p)$ satisfies the following very important relation:*

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}.$$

Proof Using the point two of the last theorem, we obtain

$$\Gamma(p)\Gamma(1-p) = \Gamma(p+1-p)\beta(p,1-p) = \Gamma(1)\beta(p,1-p) =$$

$$= \int_0^1 x^{p-1}(1-x)^{-p}\mathrm{d}x = \int_0^1 \frac{x^{p-1}}{(1-x)^p}\mathrm{d}x = \int_0^1 \left(\frac{x}{1-x}\right)^p \frac{1}{x}\mathrm{d}x.$$

We now introduce the variable y by

$$\left.\frac{x}{1-x}\right|_0^1 = y|_0^\infty \Rightarrow x = \frac{y}{1+y} \Rightarrow \mathrm{d}x = \frac{y}{(1+y)^2}\mathrm{d}y.$$

Thus for the product $\Gamma(p)\Gamma(1-p)$ we find the form

$$\Gamma(p)\Gamma(1-p)=\int_0^\infty y^p\frac{1+y}{y}\frac{1}{(1+y)^2}\mathrm{d}y=\int_0^\infty \frac{y^{p-1}}{1+y}\mathrm{d}y.$$

Now, we use the residues theorem to compute this integral. As we already know,

$$I=\int_0^\infty \frac{y^{p-1}}{1+y}\mathrm{d}y=\frac{2\pi i}{1-e^{2p\pi i}}\mathrm{res}(f,-1),\ \text{where}\ f(z)=\frac{z^{p-1}}{1+z}\Rightarrow$$

$$\Rightarrow \mathrm{res}(f,-1)=(-1)^{p-1}\Rightarrow I=\frac{(-1)^{p-1}2\pi i}{1-e^{2p\pi i}}=\frac{(-1)^p 2\pi i}{e^{2p\pi i}-1}.$$

But, using the formula for u^v and the definition of the complex logarithmic function, we have

$$(-1)^p=e^{p\ln(-1)}=e^{p(\ln 1+\pi i)}=e^{p\pi i}.$$

Finally, we obtain

$$\Gamma(p)\Gamma(1-p)=\frac{e^{p\pi i}2\pi i}{e^{2p\pi i}-1}=\frac{2\pi i}{e^{p\pi i}-e^{-p\pi i}}=\frac{2\pi i}{2i\sin p\pi}=\frac{\pi}{\sin p\pi}.$$

The proof of the theorem is closed. ∎

Application. By using the complements formula, we compute $\Gamma(1/2)$. Thus

$$z=\frac{1}{2}\Rightarrow \Gamma^2\left(\frac{1}{2}\right)=\pi\Rightarrow\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

On the other hand, if we extend the property $\Gamma(1+n)=n!$, we obtain

$$\left(\frac{1}{2}\right)!=\Gamma\left(1+\frac{1}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}\Rightarrow\left(\frac{1}{2}\right)!=\frac{\sqrt{\pi}}{2}.$$

In the final part of this paragraph we indicate a new form for the Euler's function of second species.

Theorem 2.1.4 *The function $\beta(p,q)$ can be written in the form*

$$\beta(p,q)=\int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\mathrm{d}x=\int_0^\infty\frac{x^{q-1}}{(1+x)^{p+q}}\mathrm{d}x.$$

Proof We use the definition of the function $\beta(p,q)$

$$\beta(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t,$$

and introduce a new variable x by

$$\left.\frac{t}{1-t}\right|_0^1 = x|_0^\infty \Rightarrow t=\frac{x}{1+x} \Rightarrow \mathrm{d}t=\frac{1}{(1+x)^2}\mathrm{d}x.$$

Thus the function $\beta(p,q)$ becomes

$$\beta(p,q)=\int_0^\infty \left(\frac{x}{1+x}\right)^{p-1}\left(1-\frac{x}{1+x}\right)^{q-1}\frac{1}{(1+x)^2}\mathrm{d}x=$$

$$=\int_0^\infty \frac{x^{p-1}}{(1+x)^{p-1}}\frac{1}{(1+x)^{q-1}}\frac{1}{(1+x)^2}\mathrm{d}x=\int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\mathrm{d}x.$$

The theorem is concluded. ■

Application. As an application of the last theorem, we compute the following integral

$$I=\int_0^\infty \frac{\sqrt[4]{x}}{(1+x)^2}\mathrm{d}x.$$

It is easy to see that we can write the integral in the form

$$I=\int_0^\infty \frac{x^{1/4}}{(1+x)^2}\mathrm{d}x=\int_0^\infty \frac{x^{5/4-1}}{(1+x)^{5/4+3/4}}\mathrm{d}x=\beta\left(\frac{5}{4},\frac{3}{4}\right).$$

Now, we use the connection between the functions Γ and β and, then the complements formula

$$\beta\left(\frac{5}{4},\frac{3}{4}\right)=\frac{\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})}{\Gamma(2)}=\Gamma\left(\frac{1}{4}+1\right)\Gamma\left(1-\frac{1}{4}\right)=$$

$$=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)=\frac{1}{4}\frac{\pi}{\sin\pi/4}=\frac{\pi\sqrt{2}}{4}.$$

2.2 Bessel's Functions

Consider the differential equation

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0, \tag{2.1}$$

where the unknown function is y of variable x, $y = y(x)$. Also, p is a complex parameter. This equation is called *the Bessel's equation*.

Definition 2.2.1 By definition, the solutions of Eq. (2.1) are called the Bessel's functions and are denoted by $J_p(x)$ and $J_{-p}(x)$.

We intend to give an explicit form of the solutions of Eq. (2.1), i.e. of the Bessel's functions.

Theorem 2.2.1 *The functions $J_p(x)$ and $J_{-p}(x)$ have the following polynomial form*

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m},$$

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-p+1)} \left(\frac{x}{2}\right)^{2m},$$

where Γ is the Euler's function of first species.

Proof We are looking for the solution of Eq. (2.1) in the form of an infinite polynomial

$$y(x) = x^r \sum_{k=0}^{\infty} C_k x^k. \tag{2.2}$$

We must find the constant r and the coefficients C_k such that the function $y(x)$ from Eq. (2.2) verifies Eq. (2.1). By direct calculations, we obtain

$$y(x) = \sum_{k=0}^{\infty} C_k x^{k+r} \Rightarrow y'(x) = \sum_{k=0}^{\infty} (k+r) C_k x^{k+r-1} \Rightarrow$$

$$\Rightarrow y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) C_k x^{k+r-2}.$$

If we introduce these derivatives in the Bessel's equation, it follows

$$x^r \left[\sum_{k=0}^{\infty}(k+r)(k+r-1)C_k x^k + \sum_{k=0}^{\infty}(k+r)C_k x^k + \sum_{k=0}^{\infty} C_k x^{k+2} - p^2 \sum_{k=0}^{\infty} C_k x^{k+1}\right] = 0.$$

So, we can write

$$\sum_{k=0}^{\infty}\left[(k+r)(k+r-1)+(k+r)-p^2\right]C_k x^k=-\sum_{k=0}^{\infty}C_k x^{k+2}.$$

By identifying the coefficients, it results

$$k=0 \Rightarrow \left[r(r-1)-p^2\right]C_0=0 \Rightarrow r=\pm p$$

$$k=1 \Rightarrow \left[(r+1)r+r+1-p^2\right]C_1=0 \Rightarrow \left[(r+1)^2-p^2\right]C_1=0 \Rightarrow C_1=0$$

$$k=2 \Rightarrow \left[(r+2)(r+1)+r+2-p^2\right]C_2=-C_0 \Rightarrow$$

$$\Rightarrow \left[(r+2)^2-p^2\right]C_2=-C_0 \Rightarrow 1.4(r+1)C_2=-C_0$$

$$k=3 \Rightarrow \left[(r+3)(r+2)+r+3-p^2\right]C_3=-C_1 \Rightarrow C_3=0 \text{ (because } C_1=0).$$

Thus, we deduce that $C_{2k+1}=0,\ \forall k \in N$ and

$$\begin{aligned} 1.4(p+1)C_2 &= -C_0 \\ 2.4(p+2)C_4 &= -C_2 \\ 3.4(p+3)C_6 &= -C_4 \\ ------&---- \\ m.4(p+m)C_{2m} &= -C_{2m-2}. \end{aligned}$$

By multiplying these relations it follows

$$C_{2m}=\frac{(-1)^m C_0}{m!2^{2m}(p+1)(p+2)\dots(p+m)}.$$

The Bessel's equation is a homogeneous equation and, therefore, its solution is determined except for a constant. Thus, we are free to choose the coefficient C_0 as

$$C_0=\frac{1}{2^p\Gamma(p+1)},$$

and, thus, C_{2m} becomes

$$C_{2m}=\frac{(-1)^m}{m!2^{2m+p}(p+1)(p+2)\dots(p+m)\Gamma(p+1)}=\frac{(-1)^m}{m!2^{2m+p}\Gamma(m+p+1)}.$$

In the previous calculations we used the value $r=p$ and the solution becomes

$$y(x)=\sum_{m=0}^{\infty}C_{2m}x^{p+2m}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!2^{2m+p}\Gamma(m+p+1)}\left(\frac{x}{2}\right)^{2m+p}.$$

Therefore, for $r = p$ the Bessel's function is

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m}.$$

If we take $r = -p$ the Bessel's function becomes

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-p+1)} \left(\frac{x}{2}\right)^{2m}.$$

The theorem is proved. ∎

Application. Let us compute the function $J_{1/2}(x)$. Using the polynomial form of the Bessel's function, we obtain

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n}.$$

Using the recurrence relation of the function Γ, it follows

$$\Gamma\left(n+\frac{3}{2}\right) = \frac{3}{2}\frac{5}{2}\frac{7}{2}\dots\frac{2n+1}{2}\Gamma\left(\frac{3}{2}\right), \ \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Thus $J_{1/2}(x)$ becomes

$$J_{1/2}(x) = \frac{\sqrt{\pi}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n 3.5.7\dots(2n+1)\Gamma(n+3/2)} x^{2n} =$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

So we obtain

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

The Bessel's equation is of order two. The two above solutions determine the general solution of the equation if they are linear independent.

In the following theorem we prove that in the case of $n \notin N$ the Bessel' functions are linear independent.

Theorem 2.2.2 *If the parameter p is not natural $n \notin N$, then the functions $J_p(x)$ and $J_{-p}(x)$ are linear independent.*

Proof It is a well known the fact that a system of functions is linear independent if its Wronskian is non zero. In our case, we must prove that

$$W(x) = W\left(J_p(x), J_{-p}(x)\right) = \begin{vmatrix} J_p(x) & J_{-p}(x) \\ J_p'(x) & J_{-p}'(x) \end{vmatrix} = J_p(x)J_{-p}'(x) - J_p'(x)J_{-p}(x).$$

Let us use the fact the functions $J_p(x)$ and $J_{-p}(x)$ satisfy the Bessel's equation

$$x^2 J_p''(x) + xJ_p'(x) + \left(x^2 - p^2\right) J_p(x) = 0,$$

$$x^2 J_{-p}''(x) + xJ_{-p}'(x) + \left(x^2 - p^2\right) J_{-p}(x) = 0.$$

Multiplying the first relation by $J_{-p}(x)$ and the second by $J_p(x)$ and subtracting the resulting relations, it follows

$$x^2\left(J_p''(x)J_{-p}(x) - J_{-p}''(x)J_p(x)\right) + \left(J_p'(x)J_{-p}(x) - J_{-p}'(x)J_p(x)\right) = 0.$$

So, we obtain the following differential equation of first order

$$xW'(x) + W(x) = 0 \Rightarrow \frac{\mathrm{d}W}{W} = -\frac{\mathrm{d}x}{x} \Rightarrow W(x) = \frac{C}{x}.$$

If we prove that $C \neq 0$, it results that the Wronskian of the functions $J_p(x)$ and $J_{-p}(x)$ is non zero, such that $J_p(x)$ and $J_{-p}(x)$ are linear independent. By using the polynomial form of the Bessel's functions, we obtain

$$J_p(x) = \left(\frac{x}{2}\right)^p \frac{1}{\Gamma(p+1)} + \ldots \Rightarrow J_p'(x) = \frac{p}{2}\left(\frac{x}{2}\right)^{p-1} \frac{1}{\Gamma(p+1)} + \ldots$$

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \frac{1}{\Gamma(1-p)} + \ldots \Rightarrow J_{-p}'(x) = \frac{-p}{2}\left(\frac{x}{2}\right)^{-p-1} \frac{1}{\Gamma(1-p)} + \ldots$$

Then, the Wronskian becomes

$$W(x) = \frac{-p}{2}\left(\frac{x}{2}\right)^{-1} \frac{1}{\Gamma(p+1)\Gamma(1-p)} + \cdots + \frac{-p}{2}\left(\frac{x}{2}\right)^{-1} \frac{1}{\Gamma(p+1)\Gamma(1-p)} + \cdots =$$

$$= -\frac{2p}{p\Gamma(p)\Gamma(1-p)}\frac{1}{x} + \cdots = -\frac{2}{\Gamma(p)\Gamma(1-p)}\frac{1}{x} + \ldots$$

Comparing with the first form of W, we deduce

$$C = -\frac{2}{\Gamma(p)\Gamma(1-p)} = -\frac{2\sin p\pi}{\pi} \neq 0 \ \text{ because } \ p \notin N.$$

The theorem is concluded. ■

Remark. Because the functions $J_p(x)$ and $J_{-p}(x)$ are solutions of the Bessel's equation and are linear independent, we deduce that the general solution of the Bessel's equation is

$$y(x) = C_1 J_p(x) + C_2 J_{-p}(x),\ C_1, C_2 = \text{constants}.$$

Application 1. Let us solve the equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0.$$

Since $p = 1/2 \notin N$, we deduce that this equation has two linear independent solutions, namely $J_{1/2}(x)$ and $J_{-1/2}(x)$. Using the same way as in the case of $J_{1/2}(x)$, we obtain that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Then, the general solution of our equation is

$$y(x) = C_1 J_{\frac{1}{2}}(x) + C_2 J_{-\frac{1}{2}}(x) = C_1 \sqrt{\frac{2}{\pi x}} \sin x + C_2 \sqrt{\frac{2}{\pi x}} \cos x,$$

where C_1 and C_2 are constants.

It is easy to prove, by direct calculations, that in the case $n \in N$ the Bessel's functions $J_p(x)$ and $J_{-p}(x)$ are not linear independent. Namely, we obtain

$$J_p(x) = (-1)^n J_{-p}(x).$$

In this case, we cannot define the general solution of the Bessel's equation. To solve this problem, we introduce a new function

$$N_p(x) = \frac{\cos p\pi J_p(x) - J_{-p}(x)}{\sin p\pi},\ \ p \notin N,$$

that is called the *Neumann's function*.

It is clear that the functions $J_p(x)$ and $N_p(x)$ are linear independent since $p \notin N$ and in this case the functions $J_p(x)$ and $J_{-p}(x)$ are linear independent.

In the case when $p = n \in N$ the Neumann's functions are defined by the following limits

$$N_n(x) = \lim_{p \to n} N_p(x) = \lim_{p \to n} \frac{\cos p\pi J_p(x) - J_{-p}(x)}{\sin p\pi}.$$

By using the L'Hospital's rule, we obtain

$$N_n(x) = \lim_{p\to n} \frac{-\pi \sin p\pi J_p(x) + \cos p\pi J'_p(x) - J'_{-p}(x)}{p \cos p\pi} =$$

$$= \frac{1}{\pi}\left[\frac{\partial J_p(x)}{\partial p} - (-1)^n \frac{\partial J_{-p}(x)}{\partial p}\right]_{p=n}.$$

By direct calculations we obtain that

$$W\left(J_n(x), N_n(x)\right) = \frac{2}{\pi x} \neq 0,$$

that is, these functions are linear independent and then the general solution of the Bessel's equation is

$$y(x) = C_1 J_n(x) + C_2 N_n(x),\ \ C_1, C_2 = \text{constants}.$$

Remark. Of course, the Neumann's function is a solution of the Bessel's equation because it is a linear combination of two solution of the Bessel's equation and, the Bessel's equation is linear!

Other properties of the Bessel's functions are contained in the following theorem.

Theorem 2.2.3 *The Bessel's functions satisfy the following properties*

(i)

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x),\ \ \frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^p J_{p+1}(x).$$

(ii)

$$x J'_p(x) + p J_p(x) = x J_{p-1}(x),\ \ x J'_p(x) - p J_p(x) = -x J_{p+1}(x).$$

(iii)

$$J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x),\ \ J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x).$$

Proof **(i)** It is easy to prove these relations by using the polynomial form of the Bessel's functions.

(ii) We write $(i)_1$ in detail

$$p x^{p-1} J_p(x) + x^p J'_p(x) = x^p J_{p-1}(x) \Big| : x^{p-1} \Rightarrow$$

$$\Rightarrow p J_p(x) + x J'_p(x) = x J_{p-1}(x)$$

that is, $(ii)_1$ is proved. If we write $(i)_2$ in detail

$$-px^{-p-1}J_p(x) + x^{-p}J'_p(x) = -x^{-p}J_{p+1}(x)\Big| .x^{p-1} \Rightarrow$$

$$\Rightarrow -pJ_p(x) + xJ'_p(x) = -xJ_{p+1}(x),$$

that is, (ii)$_2$ is proved.
(iii) We add (ii)$_1$ to (ii)$_2$ and obtain

$$2xJ'_p(x) = x\left(J_{p-1}(x) - J_{p+1}(x)\right) \Rightarrow$$

$$\Rightarrow J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x),$$

that is (iii)$_1$ is proved. Subtracting (ii)$_2$ from ii)$_1$ it follows

$$2pJ_p(x) = x\left(J_{p-1}(x) + J_{p+1}(x)\right) \Rightarrow$$

$$\Rightarrow J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x),$$

that is (iii)$_2$ is proved and the theorem is concluded. ■

Application. Let us compute the Bessel's functions $J_{3/2}(x)$ and $J_{-3/2}(x)$. First, we remember that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,\ \ J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

Then

$$J'_{1/2}(x) = \left(\sqrt{\frac{2}{\pi x}}\sin x\right)' = \sqrt{\frac{2}{\pi x}}\cos x - \frac{1}{2x}\sqrt{\frac{2}{\pi x}}\sin x \Rightarrow$$

$$\Rightarrow J'_{1/2}(x) = J_{-1/2}(x) - \frac{1}{2x}J_{1/2}(x).$$

Now, we write (iii)$_1$ for $p = 1/2$:

$$J_{-1/2}(x) - J_{3/2}(x) = 2J'_{1/2}(x) \Rightarrow$$

$$\Rightarrow J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x.$$

In order to obtain $J_{-3/2}(x)$, we write (iii)$_2$ for $p = -1/2$:

$$J_{-3/2}(x) - J_{1/2}(x) = -\frac{1}{x}J_{-1/2}(x)$$

from where it results

$$J_{-3/2}(x) = J_{1/2}(x) - \frac{1}{x} J_{-1/2}(x).$$

Thus

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x - \frac{1}{x}\sqrt{\frac{2}{\pi x}} \cos x.$$

2.3 Orthogonal Polynomials

Consider a sequence of functions $\{f_n\}_{n \in N}$ such that

$$f_n : [a, b] \to R, \ f_n \in C^0[a, b], \ \forall n \in N,$$

and the function

$$p = p(x), \ p : [a, b] \to R_+, \ p \in L[a, b],$$

called *the weight function*.

Definition 2.3.1 The real number denoted by (f_n, f_m) and defined by

$$(f_n, f_m) = \int_a^b p(x) f_n(x) f_m(x) \mathrm{d}x, \tag{2.3}$$

is called the scalar product of the functions f_n and f_m.

It is easy to prove the usual properties of a scalar product.

Proposition 2.3.1 *The real scalar product (3.1) has the following properties*

(i) $(f_n, f_m) = (f_m, f_n)$;
(ii) $(\lambda f_n, f_m) = \lambda (f_m, f_n)$;
(iii) $(f_n, f_m + f_k) = (f_n, f_m) + (f_n, f_k)$.

Proof All these properties are obtained based on the respective properties of the integral. For instance,

$$(f_n, f_m + f_k) = \int_a^b p(x) f_n(x) \left[f_m(x) + f_k(x) \right] \mathrm{d}x =$$

$$= \int_a^b p(x) f_n(x) f_m(x) \mathrm{d}x + \int_a^b p(x) f_n(x) f_k(x) \mathrm{d}x = (f_n, f_m) + (f_n, f_k) \,.$$

The readers can easily prove the other properties. ■

Remark. Based on the above properties of the scalar product, we deduce
(iv) $(f_n, \lambda f_m) = \lambda (f_m, f_n)$;
(v) $(f_n + f_m, f_k) = (f_n, f_k) + (f_m, f_k)$.

Definition 2.3.2 A sequence of functions $\{f_n\}_{n\in N}$ is called orthogonal if

$$(f_n, f_m) = \begin{cases} 0, & \text{for } n \neq m \\ c_n > 0, & \text{for } n = m. \end{cases}$$

If we take $f_m = f_n$ in the definition of the scalar product, then

$$(f_n, f_m) = \int_a^b p(x) f_n^2(x)\mathrm{d}x = \|f_n\|^2 \Rightarrow \|f_n\| = \sqrt{(f_n, f_n)}.$$

So, in the definition of a orthogonal sequence, we can take $c_n = \|f_n\|^2$.

Definition 2.3.3 A sequence of functions $\{f_n\}_{n\in N}$ is called orthonormal if

$$(f_n, f_m) = \begin{cases} 0, & \text{for } n \neq m, \\ 1, & \text{for } n = m. \end{cases}$$

In other words, an orthonormal sequence is an orthogonal sequence whose every element has unit norm, i.e. $\|f_n\| = 1, \ \forall n \in N$.

The following two propositions establish the connection between an orthogonal system of functions and a linearly independent system of functions.

Proposition 2.3.2 *Any orthogonal system of functions is a linear independent system of functions.*

Proof Consider the following orthogonal system of functions

$$\{f_1, f_2, \ldots, f_n\},$$

and a linear combination that is null:

$$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0.$$

In order to prove that our system is linear independent, we must prove that $\alpha_k = 0, \ k = 1, 2, \ldots, n$. By multiplying, both members of the above combinations, scalarly by f_k and using the linearity of the scalar product we obtain

$$(f_k, \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n) = 0 \Rightarrow$$

$$\Rightarrow \alpha_1 (f_k, f_1) + \alpha_2 (f_k, f_2) + \cdots + \alpha_k (f_k, f_k) + \cdots + \alpha_n (f_k, f_n) = 0.$$

Since $(f_k, f_n) = 0, \forall n \neq k$ and $(f_k, f_k) > 0$, we deduce that $\alpha_k = 0$ and the proposition is concluded. ■

Proposition 2.3.3 *From every linear independent system of functions one can extract extract an orthogonal system of functions.*

Proof Consider the following linear independent system of fucntions

$$\{f_1, f_2, \ldots, f_n\},$$

and construct the system $\{g_1, g_2, \ldots, g_n\}$ as follows:
(1) define $g_1 = f_1$;
(2) define $g_2 = f_2 + \lambda_1 g_1$ such that $(g_2, g_1) = 0$. So we find

$$\lambda_1 = -\frac{(f_2, g_1)}{(g_1, g_1)}.$$

(3) define $g_3 = f_3 + \lambda_2 g_2 + \lambda_1 g_1$ such that $(g_3, g_1) = 0$ and $(g_3, g_2) = 0$. So we find

$$\lambda_1 = -\frac{(f_3, g_1)}{(g_1, g_1)}, \ \lambda_2 = -\frac{(f_3, f_2)}{(g_2, g_2)}.$$

(4) in the general case we define $g_n = f_n + \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_{n-1} g_{n-1}$ such that $(g_n, g_1) = 0$, $(g_n, g_2) = 0$,...,$(g_n, g_{n-1}) = 0$. So we find

$$\lambda_1 = -\frac{(f_n, g_1)}{(g_1, g_1)}, \ \lambda_2 = -\frac{(f_n, g_2)}{(g_2, g_2)}, \ldots, \lambda_{n-1} = -\frac{(f_n, g_{n-1})}{(g_{n-1}, g_{n-1})}.$$

The proposition is proved.

Remark. It is easy to see that this is the *Gram–Schmidt orthogonalization procedure*.

In the following proposition we indicate, without proof, two properties of a orthogonal system of polynomials.

Proposition 2.3.4 *In the case of a orthogonal system of polynomials, we have:*

(i) For any orthogonal polynomial all its roots are real, distinct and lying in the interval of definition.

(ii) For any orthogonal polynomial we have the following recurrence relation

$$P_n(x) = (A_n x + B_n)\, P_{n-1}(x) + C_n P_{n-2}(x),$$

where A_n, B_n and C_n are constants.

Remark. If in the definition of an orthogonal polynomial we particularize the weight function and the interval of definition, we obtain different kinds of polynomials, as follows

(1) $[a, b] \to [-1, 1],\ p(x) = 1 \Rightarrow$ Legendre's polynomial, $P_n(x)$;
(2) $[a, b] \to (-1, 1),\ p(x) = 1/\sqrt{1-x^2} \Rightarrow$ Chebyshev's polynomial, $T_n(x)$;
(3) $[a, b] \to (-\infty, \infty),\ p(x) = e^{-x^2} \Rightarrow$ Hermite's polynomial, $H_n(x)$;
(4) $[a, b] \to [0, \infty),\ p(x) = e^{-x} \Rightarrow$ Laguerre's polynomial, $L_n(x)$.

2.4 Legendre's Polynomials

First, we remember two well known binomial series

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}x^n + \ldots$$

$$(1-x)^\alpha = 1 - \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 - \cdots + (-1)^n \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}x^n + \ldots$$

Consider the function

$$f(x, r) = \frac{1}{\sqrt{1-2xr+r^2}},\ |r| < 1,$$

that is called the generating function of the Legendre's polynomials.
We expand this function as a power of r series, having the coefficients as functions of x. By definition, the coefficients of this series are the Legendre's polynomials. Let us obtain the form of the Legendre's polynomials.

Theorem 2.4.1 *The expression of the Legendre's polynomials is*

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k C_{n-k}^k \frac{1.3.5....(2n-2k-1)}{(n-k)!2^k} x^{n-2k}. \tag{2.4}$$

Proof We shall begin with the power of r series of the generating function

$$\frac{1}{\sqrt{1-2xr+r^2}} = \sum_{n=0}^{\infty} P_n(x) r^n.$$

Denote $2xr - r^2 = u$, the above binomial series becomes

$$\frac{1}{\sqrt{1-u}} = (1-u)^{-1/2} = 1 - \frac{-1/2}{1!}u + \frac{-1/2(-1/2-1)}{2!}u^2 - \cdots =$$

$$= 1 + \frac{1}{2.1!}u + \frac{1.3}{2^2.2!}u^2 + \frac{1.3.5}{2^3.3!}u^3 + \cdots + \frac{1.3.5\ldots(2k-1)}{2^k.k!}u^k + \ldots$$

Thus

$$\frac{1}{\sqrt{1-2xr+r^2}} = 1 + \frac{1}{2.1!}\left(2xr - r^2\right) + \frac{1.3}{2^2.2!}\left(2xr - r^2\right)^2 + \cdots +$$

$$+\frac{1.3.5\ldots(2k-1)}{2^k.k!}\left(2xr - r^2\right)^k + \ldots$$

By identifying the coefficients, we obtain, step by step the following polynomials

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \ldots,$$

and, in the general case,

$$P_n(x) = \sum_{k=0}^{[n/2]}(-1)^k C_{n-k}^k \frac{1.3.5....(2n-2k-1)}{(n-k)!2^k} x^{n-2k}.$$

The theorem is concluded.

Theorem 2.4.2 *The Legendre's polynomials satisfy the following relation, called the Olinde-Rodrigues's relation:*

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{\mathrm{d}x^n}\left(x^2-1\right)^n.$$

Proof We shall begin with the equality

$$\left(x^2-1\right)^n = \sum_{k=0}^{n}(-1)^k C_n^k x^{2n-2k},$$

from where, by derivative,

$$\frac{d^n}{\mathrm{d}x^n}\left(x^2-1\right)^n = \sum_{k=0}^{[n/2]}(-1)^k C_n^k (2n-2k)(2n-2k-1)\ldots(n-2k+1)x^{n-2k}.$$

We now make some estimations on the coefficient of x^{n-2k}:

$$\frac{n!}{k!(n-k)!}\frac{(2n-2k)!}{(n-2k)!} = \frac{n!(2n-2k)!}{[(n-k)!]^2}\frac{(n-k)!}{k!(n-2k)!} =$$

$$= n!C_{n-k}^k \frac{1.3.5\ldots(2n-2k-1).2.4\ldots2(n-k)}{[(n-k)!]^2} = n!2^n C_{n-k}^k \frac{1.3.5\ldots(2n-2k-1)}{(n-k)!2^k},$$

from where we deduce

$$\frac{d^n}{\mathrm{d}x^n}\left(x^2-1\right)^n = n!2^n P_n(x),$$

that is, the Olinde-Rodrigues's relation and the theorem is concluded. ■

Theorem 2.4.3 *The Legendre's polynomials satisfy the following recurrence relation*

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,\ \ n = 1, 2, \ldots$$

Proof We shall begin with the equality

$$\frac{1}{\sqrt{1-2xr+r^2}} = \sum_{n=0}^{\infty} P_n(x)r^n.$$

By derivation with regard to r, it follows

$$\frac{x-r}{\sqrt{1-2xr+r^2}} = \left(1-2xr+r^2\right)\sum_{n=0}^{\infty} nP_n(x)r^{n-1} \Rightarrow$$

$$\Rightarrow (x-r)\sum_{n=0}^{\infty} P_n(x)r^n = \left(1-2xr+r^2\right)\sum_{n=0}^{\infty} nP_n(x)r^{n-1}.$$

In the last equality we identify the coefficient of r^n from both sides of the equality:

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x) \Rightarrow$$

$$\Rightarrow (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

The proof of the theorem is closed. ■

Remark. By using the recurrence relation we can determine step by step the Legendre's polynomials, starting with P_0, P_1, and so on.

Theorem 2.4.4 *The Legendre's Polynomials satisfy the following differential equation:*

$$\left(x^2-1\right)y''(x) + 2xy'(x) - n(n+1)y(x) = 0.$$

Proof We shall begin with the equality

$$\left(x^2-1\right)\frac{d}{\mathrm{d}x}\left(x^2-1\right)^n = 2nx\left(x^2-1\right).$$

By using the Leibniz's rule for derivative of superior order, we obtain

$$\left[\left(x^2-1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-1\right)^n\right]^{(n+1)}=2n\left[x\left(x^2-1\right)\right]^{(n+1)}.$$

By direct calculations

$$\left(x^2-1\right)\frac{\mathrm{d}^{n+2}}{\mathrm{d}x^{n+2}}\left(x^2-1\right)^n+2(n+1)x\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(x^2-1\right)^n+n(n+1)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n=$$

$$=2n\left[x\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(x^2-1\right)^n+(n+1)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n\right]\Rightarrow$$

$$\Rightarrow\left(x^2-1\right)\frac{\mathrm{d}^{n+2}}{\mathrm{d}x^{n+2}}\left(x^2-1\right)^n+2x\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(x^2-1\right)^n-n(n+1)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n=0.$$

We can write the relation in the form

$$\left(x^2-1\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n\right]+2x\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n\right]-n(n+1)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n=0.$$

We multiply the both sides of the last equality by $1/2^n n!$:

$$\left(x^2-1\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n\right]+2x\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n\right]-$$

$$-n(n+1)\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n=0,$$

such that, by using the Olinde-Rodrigues's relation, we obtain

$$\left(x^2-1\right)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0.$$

The theorem is concluded. ■

Theorem 2.4.5 *The Legendre's polynomials satisfy the following orthogonality relation:*

$$\int_{-1}^{1}P_n(x)P_m(x)\mathrm{d}x=\begin{cases}0, & \text{if } n\neq m\\ \frac{2}{2n+1}, & \text{if } n=m.\end{cases}$$

Proof We use the fact that the Legendre's polynomials P_n and P_m satisfy its differential equations:

$$\left(x^2-1\right)P_n''(x)+2xP_n'(x)-n(n+1)P_n(x)=0,$$

$$\left(x^2-1\right)P_m''(x)+2xP_m'(x)-m(m+1)P_m(x)=0.$$

Now, we multiply the first equation by P_m and the second by P_n and then subtracting the resulting relations, it follows

$$\left(x^2-1\right)\left(P_n''(x)P_m(x)-P_m''(x)P_n(x)\right)+2x\left(P_n'(x)P_m(x)-P_m'(x)P_n(x)\right)-$$
$$-\left(n^2+n-m^2-m\right)P_n(x)P_m(x)=0,$$

and this relation can be, equivalently, written

$$\left(x^2-1\right)\left(P_n'(x)P_m(x)-P_m'(x)P_n(x)\right)'+\left(x^2-1\right)'\left(P_n'(x)P_m(x)-P_m'(x)P_n(x)\right)=$$
$$=(n-m)(n+m+1)P_n(x)P_m(x),$$

or,

$$\left[\left(x^2-1\right)\left(P_n'(x)P_m(x)-P_m'(x)P_n(x)\right)\right]'=(n-m)(n+m+1)P_n(x)P_m(x).$$

Now, we integrate this equality

$$\int_{-1}^{1}\left[\left(x^2-1\right)\left(P_n'(x)P_m(x)-P_m'(x)P_n(x)\right)\right]'\mathrm{d}x=$$
$$=(n-m)(n+m+1)\int_{-1}^{1}P_n(x)P_m(x)\mathrm{d}x.$$

The left-hand side term of the last equality is null such that we obtain

$$(n-m)(n+m+1)\int_{-1}^{1}P_n(x)P_m(x)\mathrm{d}x=0.$$

If we suppose that $n\neq m$ it results

$$\int_{-1}^{1}P_n(x)P_m(x)\mathrm{d}x=0,$$

that is, the first relation of the theorem.

In the case $n=m$ we use the recurrence relation of the Legendre's polynomials, written for P_n and P_{n-1} and multiply the first relation by P_{n-1} and the second by P_n:

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0|\,.P_{n-1},$$

$$nP_n(x)-(2n-1)xP_{n-1}(x)+(n-1)P_{n-2}(x)=0|\,.P_n\Rightarrow$$

$$\Rightarrow nP_{n-1}^2(x) = (2n+1)xP_n(x)P_{n-1}(x) - (n+1)P_{n+1}(x)P_{n-1}(x),$$

$$nP_n^2(x) = (2n-1)xP_n(x)P_{n-1}(x) - (n-1)P_n(x)P_{n-2}(x).$$

We shall integrate these relations. By using the first part of the proof, it follows

$$\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)\mathrm{d}x = 0,$$

$$\int_{-1}^{1} P_n(x)P_{n-2}(x)\mathrm{d}x = 0,$$

such that we obtain

$$n\int_{-1}^{1} P_{n-1}^2(x)\mathrm{d}x = (2n+1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\mathrm{d}x,$$

$$n\int_{-1}^{1} P_n^2(x)\mathrm{d}x = (2n-1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\mathrm{d}x, \Rightarrow$$

$$\Rightarrow \int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)\mathrm{d}x.$$

If we use the notation

$$I_n = \int_{-1}^{1} P_n^2(x)\mathrm{d}x,$$

then

$$I_n = \frac{2n-1}{2n+1}I_{n-1}.$$

We write this relation for $n = 1, 2, 3, \ldots$, multiply the resulting relations, such that after the simplification, it follows

$$I_n = \frac{3}{2n+1}I_1.$$

But, for $n = 1$ the Legendre's polynomial is $P_1(x) = x$ such that

$$I_1 = \int_{-1}^{1} x^2 \mathrm{d}x = \frac{2}{3},$$

and then

$$I_n = \frac{3}{2n+1}\frac{2}{3} = \frac{2}{2n+1}.$$

Finally,

$$\int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2}{2n+1},$$

and the theorem is proved. ■

2.5 Chebyshev's Polynomials

Definition 2.5.1 The functions of the form

$$T_n(x) = \cos(n \arccos x),$$

are called the Chebyshev's polynomials.

Theorem 2.5.1 *The Chebyshev's polynomials have the following expression*

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k C_n^{2k} x^{n-2k} \left(1 - x^2\right)^k.$$

Proof We shall begin with the Moivre's formula

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n = C_n^0 \cos^n\theta + iC_n^1 \cos^{n-1}\theta \sin\theta -$$

$$-C_n^2 \cos^{n-2}\theta \sin^2\theta - iC_n^3 \cos^{n-3}\theta \sin^3\theta + C_n^4 \cos^{n-4}\theta \sin^4\theta + \dots$$

Now we equalize the real parts of both sides of this relation:

$$\cos n\theta = C_n^0 \cos^n\theta - C_n^2 \cos^{n-2}\theta \sin^2\theta + C_n^4 \cos^{n-4}\theta \sin^4\theta + \dots \Rightarrow$$

$$\Rightarrow \cos n\theta = \sum_{k=0}^{n} (-1)^k C_n^{2k} \cos^{n-2k}\theta \sin^{2k}\theta.$$

With the aid of the substitution

$$x = \cos\theta \Rightarrow \theta = \arccos x$$

we obtain

$$\cos(n \arccos x) = \sum_{k=0}^{[n/2]} (-1)^k C_n^{2k} x^{n-2k} \left(1 - x^2\right)^k,$$

such that

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k C_n^{2k} x^{n-2k} \left(1 - x^2\right)^k,$$

such that the proof is closed. ■

Theorem 2.5.2 *The Chebyshev's polynomials have the function*

$$f(x) = \frac{1 - rx}{1 - 2xr + r^2},$$

as generating function.

Proof We expand this function as a power series of r and prove that the coefficients of this series are the Chebyshev's polynomials, that is

$$\frac{1 - rx}{1 - 2xr + r^2} = \sum_{n=0}^{\infty} T_n(x) r^n.$$

If we use the substitution $x = \cos\theta$ it follows

$$\frac{1 - rx}{1 - 2xr + r^2} = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + \cos^2\theta} =$$

$$= \frac{1 - r\cos\theta}{(r - \cos\theta - i\sin\theta)(r - \cos\theta + i\sin\theta)} = \frac{1 - r\cos\theta}{(r - e^{i\theta})(r - e^{-i\theta})} =$$

$$= \frac{1 - r\cos\theta}{(1 - re^{i\theta})(1 - re^{-i\theta})} = \frac{1}{2}\left(\frac{1}{1 - re^{i\theta}} + \frac{1}{1 - re^{-i\theta}}\right) =$$

$$= \frac{1}{2}\left(1 + re^{i\theta} + r^2 e^{2i\theta} + \cdots + r^n e^{ni\theta} + \cdots + 1 + re^{-i\theta} + r^2 e^{-2i\theta} + \cdots + r^n e^{-ni\theta} + \cdots\right) =$$

$$= 1 + r\frac{e^{i\theta} + e^{-i\theta}}{2} + r^2\frac{e^{2i\theta} + e^{-2i\theta}}{2} + \cdots + r^n\frac{e^{ni\theta} + e^{-ni\theta}}{2} + \cdots =$$

$$= 1 + r\cos\theta + r^2\cos 2\theta + \cdots + r^n\cos n\theta + \cdots = \sum_{n=0}^{\infty} r^n \cos n\theta.$$

If we substitute $\theta = \arccos x$, it follows that the coefficients of the series are

$$\cos n\theta = \cos(n \arccos x) = T_n(x),$$

and the theorem is proved. ■

Remark. The Chebyshev's polynomials do not satisfy an Olinde-Rodrigues's relation.

Theorem 2.5.3 *The Chebyshev's polynomials satisfy the following recurrence relation*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Proof We shall begin with the formula

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta \cos n\theta.$$

If we substitute here θ by $\arccos x$ it results

$$\cos((n+1)\arccos x) + \cos((n-1)\arccos x) = 2x\cos(n \arccos x),$$

that is,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

The proof of the theorem is closed. ■

Remark. By using the recurrence relation, we can determine step by step the Chebyshev's polynomials, starting with P_0, P_1, and so on.

Theorem 2.5.4 *The Chebyshev's polynomials satisfy the following differential equation:*

$$\left(1 - x^2\right) y''(x) - xy'(x) + n^2 y(x) = 0.$$

Proof We derive with regard to x the relation of definition for the Chebyshev's polynomials:

$$T_n(x) = \cos(n \arccos x)|_x' \Rightarrow$$

$$\Rightarrow T_n'(x) = -n\sin(n \arccos x)\frac{-1}{\sqrt{1-x^2}}.$$

This relation can be written in the form

$$\sqrt{1-x^2}T_n'(x) = n\sin(n \arccos x).$$

We derive here both sides with regard to x

$$\frac{-x}{\sqrt{1-x^2}}T_n'(x) + \sqrt{1-x^2}T_n''(x) = n^2\cos(n \arccos x)\frac{-1}{\sqrt{1-x^2}}.$$

Multiplying both sides by $-\sqrt{1-x^2}$, it follows

$$xT_n'(x) - \left(1-x^2\right)T_n''(x) = n^2T_n(x) \Rightarrow$$

$$\Rightarrow \left(1-x^2\right)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0.$$

The theorem is concluded. ■

Theorem 2.5.5 *The Chebyshev's polynomials satisfy the following orthogonality relation:*

$$(T_n, T_m) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)T_m(x)\mathrm{d}x = \begin{cases} 0, \text{ if } n \neq m \\ \frac{\pi}{2}, \text{ if } n = m \neq 0 \\ \pi, \text{ if } n = m = 0 \end{cases}$$

because the weight function and the interval of definition for the Chebyshev's polynomials are

$$p(x) = \frac{1}{\sqrt{1-x^2}}, \ [a, b] \to (-1, 1).$$

Proof We substitute the expressions of T_n and T_m in the definition of the scalar product

$$(T_n, T_m) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)T_m(x)\mathrm{d}x =$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cos(n \arccos x)\cos(m \arccos x)\mathrm{d}x.$$

Here, we substitute x by $\cos\theta$

$$x = \cos\theta \Rightarrow \mathrm{d}x = -\sin\theta\mathrm{d}\theta;\, x \in (-1, 1) \Rightarrow \theta \in (\pi, 0).$$

Then, we obtain

$$(T_n, T_m) = \int_{\pi}^{0} \frac{1}{\sin\theta} \cos n\theta \cos m\theta(-\mathrm{d}\theta)\sin\theta =$$

$$= \int_{0}^{\pi} \cos n\theta \cos m\theta \mathrm{d}\theta = \frac{1}{2}\int_{0}^{\pi} [\cos(n+m)\theta + \cos(n-m)\theta\mathrm{d}\theta.$$

(i) If $n \neq m$ then

$$(T_n, T_m) = \frac{1}{2}\left[\frac{\sin(n+m)\theta}{n+m}\bigg|_0^\pi + \frac{\sin(n-m)\theta}{n-m}\bigg|_0^\pi\right] = 0.$$

(ii) If $n = m \neq 0$ then

$$(T_n, T_m) = \frac{1}{2}\int_0^\pi (\cos 2n\theta + 1)\mathrm{d}\theta = \frac{1}{2}\left[\frac{\sin 2n\theta}{2n}\bigg|_0^\pi + \theta|_0^\pi = \frac{\pi}{2}\right].$$

(iii) If $n = m = 0$ then

$$(T_n, T_m) = \frac{1}{2}\int_0^\pi 2\mathrm{d}\theta = \int_0^\pi \mathrm{d}\theta = \theta\bigg|_0^\pi = \pi.$$

So, the theorem is proved. ■

2.6 Hermite's Polynomials

Definition 2.6.1 The functions defined by

$$H_n(x) = e^{x^2}\frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right), \tag{2.5}$$

are called the Hermite's polynomials.

Remark. As it is easy to see, the Hermite's polynomials are defined direct by an Olinde-Rodrigues's relation.

The first Hermite's polynomials are

$$H_0(x) = e^{x^2}\left(e^{-x^2}\right)^{(0)} = e^{x^2}e^{-x^2} = 1,$$

$$H_1(x) = e^{x^2}\left(e^{-x^2}\right)' = e^{x^2}\left(-2xe^{-x^2}\right) = -2x,$$

$$H_2(x) = e^{x^2}\left(e^{x^2}\right)'' = e^{x^2}\left(-2xe^{-x^2}\right)' = -2xe^{x^2}\left(e^{-x^2} - 2x^2e^{-x^2}\right) = 4x^2-2, \ldots$$

The generating function of the Hermite's polynomials is the function

$$h(r, x) = e^{-(r^2+2xr)},$$

that is, if we expand as a power series of r this function, the coefficients of the series are the Hermite's polynomials:

$$e^{-(r^2+2xr)} = \sum_{n=0}^{\infty} H_n(x) \frac{1}{n!} r^n$$

Theorem 2.6.1 *The Hermite's polynomials satisfy the following differential equation:*

$$y''(x) - 2xy'(x) + 2ny(x) = 0.$$

Proof We derive with regard to x both sides of the relation (6.1)

$$H_n'(x) = \left[e^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)\right]' = 2xe^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) + e^{x^2} \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right),$$

$$H_n''(x) = 2e^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) + 2x^2 e^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) + xe^{x^2} \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) +$$

$$+2xe^{x^2} \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) + e^{x^2} \frac{d^{n+2}}{\mathrm{d}x^{n+2}}\left(e^{-x^2}\right).$$

Thus

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 2e^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) +$$

$$+4x^2 e^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) + 4xe^{x^2} \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) - 4x^2 e^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) -$$

$$-2xe^{x^2} \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) + 2ne^{x^2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) + e^{x^2} \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) =$$

$$= e^{x^2} \left[2 \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) + 2x \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) + \frac{d^{n+2}}{\mathrm{d}x^{n+2}}\left(e^{-x^2}\right) + 2n \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)\right].$$

But

$$\frac{d^{n+2}}{\mathrm{d}x^{n+2}}\left(e^{-x^2}\right) = \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(-2xe^{-x^2}\right) =$$

$$= -2\left[2x \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) + (n+1) \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)\right],$$

such that, we obtain

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = e^{x^2} \left[2 \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) + 2x \frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right) -\right.$$

$$-2x\frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right)-2(n+1)\frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)+2n\frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)\Bigg]=0.$$

In conclusion, we obtain the equation

$$H_n''(x)-2xH_n'(x)+2nH_n(x)=0,$$

which is the desired equation and the theorem is proved. ∎

Theorem 2.6.2 *The Hermite's polynomials satisfy the following recurrence relation*

$$H_{n+1}(x)+2xH_n(x)+2nH_{n-1}(x)=0.$$

Proof We shall begin with the formula

$$\frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right)=\frac{d^n}{\mathrm{d}x^n}\left(-2xe^{-x^2}\right)=-2\frac{d^n}{\mathrm{d}x^n}\left(xe^{-x^2}\right).$$

Now, we apply the Leibniz's rule for derivative of superior order

$$\frac{d^n}{\mathrm{d}x^n}\left(xe^{-x^2}\right)=x\frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)+n\frac{d^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-x^2}\right),$$

$$\frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right)=-2x\frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)-2n\frac{d^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-x^2}\right)\Rightarrow$$

$$\Rightarrow e^{x^2}\frac{d^{n+1}}{\mathrm{d}x^{n+1}}\left(e^{-x^2}\right)=-2xe^{x^2}\frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)-2ne^{x^2}\frac{d^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-x^2}\right).$$

Using the Olinde-Rodrigues's relation it follows

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x),$$

such that the theorem is proved. ∎

Theorem 2.6.3 *The Hermite's polynomials satisfy the following orthogonality relation:*

$$(H_n,H_m)=\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)\mathrm{d}x=\begin{cases}0, & \text{if } n\neq m\\ 2^n n!\sqrt{\pi}, & \text{if } n=m\end{cases}$$

because the weight function and the interval of definition for the Hermite's Polynomials are

$$p(x)=e^{-x^2},\ [a,b]\to(-\infty,\infty).$$

Proof Using the Olinde-Rodrigues's relation, we have

$$(H_n, H_m) = \int_{-\infty}^{\infty} e^{-x^2} H_m(x) \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) \mathrm{d}x = \int_{-\infty}^{\infty} H_m(x) \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) \mathrm{d}x =$$

$$= H_m(x) \frac{d^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-x^2}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H'_m(x) \frac{d^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-x^2}\right) \mathrm{d}x =$$

$$= -\int_{-\infty}^{\infty} H'_m(x) \frac{d^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-x^2}\right) \mathrm{d}x,$$

because we have

$$\lim_{x\to\pm\infty} P(x) e^{-x^2} = \lim_{x\to\pm\infty} \frac{P(x)}{e^{x^2}} = 0,$$

for any polynomials $P(x)$.

On the other hand, we have

$$H'_m(x) = \left[\frac{d^m}{\mathrm{d}x^m}\left(e^{-x^2}\right)\right]' = 2x e^{x^2} \frac{d^m}{\mathrm{d}x^m}\left(e^{-x^2}\right) +$$

$$+ e^{x^2} \frac{d^{m+1}}{\mathrm{d}x^{m+1}}\left(e^{-x^2}\right) = 2x H_m(x) + H_{m+1}(x),$$

such that, by using the recurrence relation, we obtain

$$H'_m(x) = -2m H_{m-1}(x).$$

Then

$$(H_n, H_m) = 2m \int_{-\infty}^{\infty} H_{m-1}(x) \frac{d^{n-1}}{\mathrm{d}x^{n-1}}\left(e^{-x^2}\right) \mathrm{d}x.$$

If we integrate one more time by parts, we are led to

$$(H_n, H_m) = 2^2(m-1)m \int_{-\infty}^{\infty} H_{m-2}(x) \frac{d^{n-2}}{\mathrm{d}x^{n-2}}\left(e^{-x^2}\right) \mathrm{d}x,$$

and, after m steps,

$$(H_n, H_m) = 2^m m! \int_{-\infty}^{\infty} \frac{d^{n-m}}{\mathrm{d}x^{n-m}}\left(e^{-x^2}\right) \mathrm{d}x.$$

If $m \neq n$ we have

$$\int_{-\infty}^{\infty} \frac{d^{n-m}}{\mathrm{d}x^{n-m}} \left(e^{-x^2}\right) \mathrm{d}x = \frac{d^{n-m-1}}{\mathrm{d}x^{n-m-1}} \left(e^{-x^2}\right)\Big|_{-\infty}^{\infty} = 0,$$

that is

$$(H_n, H_m) = 0.$$

If $m = n$ we have

$$(H_n, H_m) = 2^m m! \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = 2^m m! \sqrt{\pi},$$

and the theorem is proved. ■

2.7 Laguerre's Polynomials

Definition 2.7.1 The functions defined by

$$L_n(x) = \sum_{k=0}^{n} (-1)^k C_n^k \frac{1}{k!} x^k,$$

are called the Laguerre's polynomials.

Remark. We remember that the weight function and the interval of definition for the Laguerre's polynomials are

$$p(x) = e^{-x}, \ [a, b] \to [0, \infty).$$

In the following we indicate (without proof) the main properties of the Laguerre's polynomials. We hope that the readers can prove these results in a similar manner like the other orthogonal polynomials.

Theorem 2.7.1 *The Laguerre's polynomials satisfy the following Olinde-Rodrigues relation*

$$L_n(x) = \frac{1}{n!} \frac{d^n}{\mathrm{d}x^n} \left(x^n e^{-x}\right).$$

Theorem 2.7.2 *The generating function for the Laguerre's polynomials is*

$$f(x, r) = \frac{1}{1-r} e^{-rx/(1-r)},$$

that is,

$$\frac{1}{1-r}e^{-rx/(1-r)} = \sum_{n=0}^{\infty} L_n(x)r^n.$$

Theorem 2.7.3 *The Laguerre's polynomials are the solutions of the following differential equation*

$$xy''(x) + (1-x)y'(x) + ny(x) = 0.$$

Theorem 2.7.4 *The Laguerre's polynomials satisfy the following recurrence relation*

$$(n+1)L_{n+1}(x) - (2n+1-x)L_n(x) + nL_{n-1}(x) = 0, \ \forall n \in N.$$

Theorem 2.7.5 *The Laguerre's polynomials satisfy the following orthogonality relation:*

$$(L_n, L_m) = \int_0^{\infty} e^{-x} L_n(x) L_m(x) \mathrm{d}x = \begin{cases} 0, \ \text{if } n \neq m \\ (n!)^2, \ \text{if } n = m. \end{cases}$$

One of the main reasons for the study of special functions is to solve certain differential equations with variable coefficients. In this sense, we shall consider a few applications.

Applications 1. Find the solutions of the equation

$$x^2y'' + 2xy' + \left(x^3 - 1\right)y = 0, \ y = y(x).$$

Indications. First, we make the change of variable

$$x^3 = t^2.$$

Then, we pass to a new unknown function u with the transformation

$$y(t) = u(t)t^{\alpha},$$

and determine α such that the coefficient of u' is t as in the standard form of a Bessel's equation. Finally, we make a new change of variable:

$$\frac{2}{3}t = \tau,$$

such that the equation received the standard form of a Bessel's equation.

2. Find a solution of the equation

$$\left(x^2 - 1\right)y'' + 2xy' - 6y = 0, \ y = y(x).$$

Solution. It is easy to remark that we have a Legendre equation in the particular case $n = 2$ such that a solution is

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

3. Find a solution of the equation

$$\left(1 - x^2\right) y'' - xy' + 9y = 0, \ \ y = y(x).$$

Solution. It is easy to remark that we have a Chebyshev equation in the particular case $n = 3$ such that a solution is

$$T_3(x) = \cos(3\arccos x).$$

4. Find a solution of the equation

$$y'' - 2xy' = 4y = 0, \ \ y = y(x).$$

Solution. It is easy to remark that we have a Hermite equation in the particular case $n = 2$ such that a solution is

$$H_2(x) = e^{x^2} \frac{d^2}{\mathrm{d}x^2} \left(e^{-x^2}\right) = -2x.$$

Chapter 3
Operational Calculus

3.1 Laplace's Transform

An useful instrument in tackling differential equations and partial differential equations is proved to be the Laplace's transform which we study in this paragraph. The Laplace's transform makes the correspondence between two functions set, one having difficult operations, and second, more accessible. For instance, a differential equation in the first functions set is transformed in an algebrical equation in the second functions set. This correspondence is made by means of a transformation. We will deal only with the Laplace's transform and Fourier's transform.

Definition 3.1.1 A function $f : \mathbb{R} \to \mathbb{R}$ is called original function for the Laplace's transform if the following conditions are satisfied:

(i). $f(t)$ and $f'(t)$ exist and are continuous throughout the real axis, possible except a sequence of points $\{t_n\}_{n\geq 1}$ in which it can appear discontinuities of first species;
(ii). $f(t) = 0, \ \forall t < 0$;
(iii). there exist the constants $M > 0, \ s_0 \geq 0$, such that

$$|f(t)| \leq Me^{s_0 t}, \ \forall t \in \mathbb{R}.$$

Usually s_0 is the *growing index* of the original. A classic example of original function is the Heaviside's function θ, defined by

$$\theta(t) = \begin{cases} 0, \text{ if } t < 0, \\ 1, \text{ if } t \geq 0. \end{cases}$$

If a function f satisfies the conditions (i) and (iii) but does not satisfy the condition (ii), from the definition of the original function, then we will make the convention that the function f is multiplied by the Heaviside's function θ:

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$$f(t) = f(t)\theta(t) = \begin{cases} 0, & \text{if } t < 0, \\ f(t), & \text{if } t \geq 0. \end{cases}$$

It is easy to see that $|f(t)| \leq 1 = 1.e^{0.t}$, such that $M = 1 > 0$ and the growing index is $s_0 = 0$.

This convention is made for increasing the set of original functions. We will denote by $\mathcal{O}$ the set of the original functions. In the following theorem we are going to prove the structure of the set $\mathcal{O}$. More exactly, we will prove that the set $\mathcal{O}$ has the structure of a linear space and even an algebra structure.

Theorem 3.1.1 *Consider $\mathcal{O}$ the set of the original functions. Then:*

1°. $f + g \in \mathcal{O}, \ \forall f, g \in \mathcal{O}$;
2°. $\lambda f \in \mathcal{O}, \ \forall f \in \mathcal{O}, \ \forall \lambda \in R$;
3°. $f.g \in \mathcal{O}, \ \forall f, g \in \mathcal{O}$.

Proof 1^o. Since $f, g \in \mathcal{O}$, we will deduce that $f + g$ satisfies obviously the properties (i) and (ii) of the original functions. Let us check the (iii) condition. If

$$|f(t)| \leq M_1 e^{s_1 t}, \ |g(t)| \leq M_2 e^{s_2 t}, \ \forall t \in R,$$

then

$$|f(t) + g(t)| \leq |f(t)| + |g(t)| \leq M_1 e^{s_1 t} + M_2 e^{s_2 t} \leq M_3 e^{s_3 t}, \ \forall t \in \mathbb{R},$$

where $s_3 = \max\{s_1, s_2\}$ and $m_3 = \max\{M_1, M_2\}$.

2^o. λf satisfies obviously the properties (i) and (ii) of the original functions. Let us check the (iii) condition. Since

$$|f(t)| \leq M_1 e^{s_1 t}, \ \forall t \in R$$

we will deduce that

$$|\lambda f(t)| = |\lambda||f(t)| \leq |\lambda| M_1 e^{s_1 t}, \ \forall t \in R,$$

that is λf has the same growing index as f.

3^o. With regard to the (iii) condition for the product $f.g$, we have

$$|f(t).g(t)| = |f(t)|.|g(t)| \leq M_1 M_2 e^{(s_1 + s_2)t}, \ \forall t \in R,$$

and the proof is over, because the properties (i) and (ii) are obvious. ■

Remarks.

1^o. From the proof of the Theorem 3.1.1, it results that

$$\text{ind}(f+g) = \max\{\text{ind}(f), \text{ind}(g)\},$$
$$\text{ind}(f.g) = \text{ind}(f) + \text{ind}(g)$$

2^o. If $f_i \in \mathcal{O},\ i = 1, 2, ..., n$, then

$$\sum_{i=1}^{n} \lambda_i f_i \in \mathcal{O},\ \forall \lambda_i \in R,\ \text{ or } \lambda_i \in \text{IC}, i = 1, 2, ..., n.$$

The statement results from the first two points of the Theorem 3.1.1.

3^o. If $f_i \in \mathcal{O},\ i = 1, 2, ..., n$, then

$$\prod_{i=1}^{n} f_i \in \mathcal{O}.$$

The statement can be immediately proved, by applying the point 3^o of the theorem. In the particular case, if $f \in \mathcal{O}$ then $f^n \in \mathcal{O},\ \forall n \in N^*$.

4^o. The function $f(t) = e^{\lambda t}$ is an original function, $\forall \lambda \in \text{IC},\ \lambda \alpha + i\beta$, having the increasing index

$$s_0 = \begin{cases} 0, & \text{if } \alpha < 0, \\ \alpha, & \text{if } \alpha \geq 0. \end{cases}$$

As a consequence, the following functions are original functions too

$$\sin \lambda t,\ \cos \lambda t,\ \sinh \lambda t,\ \cosh \lambda t.$$

If we expand the function $e^{\lambda t}$ as power series

$$e^{\lambda t} = 1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + ... + \frac{\lambda^n t^n}{n!} + ..., t \geq 0$$

and take into account the fact that

$$\frac{\lambda^n t^n}{n!} < e^{\lambda t},\ \forall t \geq 0,$$

we will immediately deduce that

$$t^n < \frac{n!}{\lambda^n t^n} e^{\lambda t},\ \forall t \geq 0,$$

and then we obtain that the function

$$f(t) = t^n,\ t \geq 0$$

is an original function.

Based on the above remarks, it follows that the function

$$f(t) = e^{\lambda t}\,[P(t)\cos \alpha t + Q(t)\sin \alpha t]$$

is an original function, for any two polynomials P and Q.

Definition 3.1.2 If $f(t)$ is an original function, with the increasing index s_0, then we call it the Laplace's transform of f, or its image through the Laplace's transform, the function F which is defined by

$$F(p) = \int_0^\infty f(t)e^{-pt}\mathrm{d}t,\ \forall p \in C,\ Re(p) \geq s_0. \tag{3.1.1}$$

Let us prove that the image function F from Eq. (3.1.1) is defined on the whole semiplan $[s_0, \infty)$ and, more, F is an analytic function in this semiplan.

Theorem 3.1.2 *If f is an original function with the increasing index s_0, then the function $F : [s_0, \infty) \to C$ has sense for any complex number p for which $Re(p) \geq s_0$ and F is an analytical function in this semiplan.*

Proof Starting from Eq. (3.1.1) we obtain

$$|F(p)| \leq \int_0^\infty \left|f(t)e^{-pt}\right|\mathrm{d}t \leq$$
$$\leq M\int_0^\infty e^{s_0 t}e^{-pt}\mathrm{d}t = \frac{M}{s-s_0}e^{(s_0-p)t}\Big|_0^\infty = \frac{M}{s-s_0},$$

inequality which proves that the function F is well defined.

If $Re(p) \geq s_1 \geq s_0$, then we can derive under the integral in Eq. (3.1.1):

$$F'(p) = \int_0^\infty -te^{-pt}\mathrm{d}t,$$

and then we find the estimations

$$|F'(p)| \leq \int_0^\infty |tf(t)|e^{-pt}\mathrm{d}t \leq$$
$$\leq M\int_0^\infty te^{(s_0-p)t}\mathrm{d}t \leq M\int_0^\infty te^{(s_0-s_1)t}\mathrm{d}t =$$
$$= Mt\frac{e^{(s_0-s_1)t}}{s_0-s_1}\Big|_0^\infty + \frac{M}{s_0-s_1}\int_0^\infty e^{(s_0-s_1)t}\mathrm{d}t = \frac{M}{(s_0-s_1)^2},$$

after that we had integrated by parts. Since the derivation is bounded, we will deduce that F is an analytical function in the open semiplan (s_0, ∞). ■

As a consequence of the Theorem 3.1.2, one can find that

$$\lim_{|p|\to\infty} |F(p)| = 0.$$

It is natural to put ourselves the question if we know a transformation F, which is the original function whose Laplace's transform is even F. The answer is given in the following theorem.

Theorem 3.1.3 *Given the Laplace's transform F, then the original function can be obtained in each point of continuity t with the aid of the Laplace's transform through of following inverse formula:*

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\,\mathrm{d}p, \tag{3.1.2}$$

where $a \in R,\ a \geq s_0$.

Because the proof is arduous, we renounce to give it.

Theorem 3.1.3 asserts that if the Laplace's transform of an original function is given, then it is the Laplace's transform of a single original, that is, the Laplace's transform is an one to one correspondence into the set of the originals. The integral on the right-hand side of Eq. (3.1.2) is an improper integral in the Cauchy sense:

$$\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\,\mathrm{d}p = \lim_{b\to\infty}\int_{a-ib}^{a+ib} F(p)e^{pt}\,\mathrm{d}p.$$

In order to outline the fact that, for a given Laplace's transform, the original function is unique determined, we prove the following theorem.

Theorem 3.1.4 *Given is the Laplace's transform F with the properties:*

1°. $F(p)$ *is an analytical function in the semiplan* $Re(p) \geq a > s_0$*;*
2°.

$$\lim_{|p|\to\infty} |F(p)| = 0,\ \ for\ \ Re(p) \geq a > s_0,$$

the limit being uniformly with regard to p;
3°. The integral

$$\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\,\mathrm{d}p$$

is absolutely convergent.

Then the function $f(t)$, *defined by*

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\mathrm{d}p, \tag{3.1.3}$$

has even the function $F(p)$ *as the Laplace's transform.*

Proof First, we make the observation that the Laplace's transform $F(p)$ will be denoted also with $\mathcal{L}(f(t))$ or, more simple, $\mathcal{L}(f)$, taking into account that the argument of the original function is denoted by t and the argument of the Laplace's transform is denoted by p.

Applying the Laplace's transform in Eq. (3.1.3):

$$\mathcal{L}(f) = \int_0^{\infty} \left\{ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\mathrm{d}p \right\} e^{-p_0 t}\mathrm{d}t. \tag{3.1.4}$$

Let us prove that $\mathcal{L}(f) = F(p_0)$, where $p_0 = a + i\sigma$ is arbitrarily fixed in the semiplan $[s_0, \infty)$.

Since

$$\left| \int_{a-i\infty}^{a+i\infty} F(p)e^{p_0 t}\mathrm{d}p \right| \le \int_{a-i\infty}^{a+i\infty} |F(p)| \left| e^{p_0 t} \right| \mathrm{d}p =$$

$$= \int_{a-i\infty}^{a+i\infty} |F(p)| \left| e^{at} \right| | \left| e^{i\sigma t} \right| \mathrm{d}p = \int_{a-i\infty}^{a+i\infty} |F(p)| \, e^{at}\mathrm{d}p,$$

and the last integral is convergent (see 3^0), we will deduce that in Eq. (3.1.4) we can commute the integrals:

$$\mathcal{L}(f) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(p) \left\{ \int_0^{\infty} e^{(p-p_0)t}\mathrm{d}t \right\} \mathrm{d}p.$$

Since $Re(p - p_0) = a - s < 0$ and $e^{(p-p_0)t}\big|_0^{\infty} = -1$, we obtain

$$\mathcal{L}(f) = \frac{1}{2\pi i} \int_{a+i\infty}^{a-i\infty} \frac{F(p)}{p - p_0}\mathrm{d}p.$$

Let us consider the circle having the origin as the center and the radius R and consider the vertical segment between $a - ib$ and $a + ib$ and the arc of circle C_R determined by this segment on the considered circle. Applying the Cauchy's formula (from the theory of complex functions), taking into account that $p = p_0$ is a singular pole, it follows:

$$F(p_0) = \frac{1}{2\pi i}\int_{a+ib}^{a-ib} \frac{F(p)}{p-p_0}\mathrm{d}p + \frac{1}{2\pi i}\int_{C_R} \frac{F(p)}{p-p_0}\mathrm{d}p. \tag{3.1.5}$$

For the last integral from Eq. (3.1.5) we have the estimation

$$\left|\frac{1}{2\pi i}\int_{C_R} \frac{F(p)}{p-p_0}\mathrm{d}p\right| \le \frac{1}{2\pi}2\pi R\frac{M_R}{|R|-|p_0|},$$

where

$$M_R = \sup_{p\in C_R} |F(p)|.$$

Based on the hypothesis 2^o we will deduce that $M_R \to 0$, for $R \to \infty$. So we deduce that the last integral from Eq. (3.1.5) converges to zero, for $R \to \infty$.

Therefore, if we take the limit in Eq. (3.1.5) for $R \to \infty$, we obtain

$$F(p_0) = \frac{1}{2\pi i}\int_{a+i\infty}^{a-i\infty} \frac{F(p)}{p-p_0}\mathrm{d}p = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{F(p)}{p-p_0}\mathrm{d}p,$$

that is $F(p_0) = \mathcal{L}(f)$. ∎

In the following proposition we will prove the main properties of the Laplace's transform.

Proposition 3.1.1 *If f and g are original functions, having the images, respectively, F and G, and $\alpha,\ \beta \in \mathbb{R}$, then*

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha F(p) + \beta G(p).$$

Proof This result can be immediately obtained, based on the linearity of the Riemann's integral. ∎

Proposition 3.1.2 *If f is an original function, having the image F, and $\alpha \in \mathbb{C}^*$, then*

$$\mathcal{L}(f(\alpha t)) = \frac{1}{\alpha}F\left(\frac{p}{\alpha}\right).$$

Proof Using the change of variable $\alpha t = \tau$, we obtain

$$\mathcal{L}(f(\alpha t)) = \int_0^\infty f(\alpha t)e^{-pt}\mathrm{d}t =$$
$$= \int_0^\infty f(\tau)e^{-\frac{p}{\alpha}\tau}\frac{1}{\alpha}\mathrm{d}\tau = \frac{1}{\alpha}\int_0^\infty f(\tau)e^{-\frac{p}{\alpha}\tau}\mathrm{d}\tau,$$

such that the result is proved. ■

Proposition 3.1.3 *If f is an original function, having the image F, then in a point t in which f is derivable, we have:*

$$\mathcal{L}(f'(t)) = pF(p) - f(0).$$

Proof Starting from the definition of the Laplace's transform, by direct calculations it follows:

$$\mathcal{L}(f'(t)) = \int_0^\infty f'(t)e^{-pt}\mathrm{d}t = e^{-pt}f(t)\Big|_0^\infty -$$
$$-\int_0^\infty (-p)f(t)e^{-pt}\mathrm{d}t = -f(0) + p\int_0^\infty f(t)e^{-pt}\mathrm{d}t = pF(p) - f(0)$$

such that the result is proved. ■

Corollary 3.1.1 *With regard to the derivative, we can prove a more general result, as follows*

$$\mathcal{L}(f^{(n)}(t)) = p^n F(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

Proof By using the Proposition 3.1.3, we have

$$\begin{aligned}\mathcal{L}(f'(t)) &= p\mathcal{L}(f(t)) - f(0),\\ \mathcal{L}(f''(t)) &= p\mathcal{L}(f'(t)) - f'(0),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \mathcal{L}(f^{(n)}(t)) &= p\mathcal{L}(f^{(n-1)}(t)) - f^{(n-1)}(0).\end{aligned}$$

Now, we multiply the first relation by p^{n-1}, the second by p^{n-2}, ..., the last by p^0. Then we add the resulting relations and we obtain the desired result. ■

Proposition 3.1.4 *If f is an original function, having the image F, then:*

$$F'(p) = \mathcal{L}(-tf(t)).$$

Proof We already proved that the integral from the definition is convergent. Then, we can derive under the integral with regard to p:

$$F(p) = \int_0^\infty f(t)e^{-pt}\mathrm{d}t \Rightarrow F'(p) = \int_0^\infty f(t)(-t)e^{-pt}\mathrm{d}t,$$

and we arrive at the desired result. ■

Corollary 3.1.2 *With regard to the derivative of the Laplace's transform we can prove a more general result, as follows*

$$F^{(n)}(p) = \mathcal{L}((-t)^n f(t)).$$

Proof This result can be easy obtained by successive derivation under the integral and then by using the mathematical induction. ■

As a consequence of this property it immediately follows that

$$\mathcal{L}(t^n) = \frac{n!}{p^{n+1}}.$$

Proposition 3.1.5 *Let f be an original function whose Laplace's transform is F. Then the integral*

$$\int_0^t f(\tau)\mathrm{d}\tau$$

is also an original function, having the same growing index like f. Moreover, the following formula is still valid

$$\mathcal{L}\left(\int_0^t f(\tau)\mathrm{d}\tau\right) = \frac{1}{p}F(p).$$

Proof It is easy to prove the (i) and (ii) conditions from the definition of an original function in the case of the integral

$$\int_0^t f(\tau)\mathrm{d}\tau,$$

taking into account that f satisfies these conditions. Let us denote by g this integral

$$g(t) = \int_0^t f(\tau)\mathrm{d}\tau.$$

We intend to prove that g satisfies the condition (iii) from the definition of an original function:

$$\begin{aligned} |g(t)| &\le \int_0^t |f(\tau)|\mathrm{d}\tau \le M\int_0^t e^{s_0\tau}\mathrm{d}\tau = \\ &= \frac{M}{s_0}\left(e^{s_0 t} - 1\right) \le \frac{M}{s_0}e^{s_0 t}, \end{aligned}$$

and it is easy to see that g has the same growing index like f.

On the other hand, since

$$g(t) = \int_0^t f(\tau)\mathrm{d}\tau,$$

it is readily seen that $g(0) = 0$ and $g'(t) = f(t)$. Therefore

$$\mathcal{L}(f(t)) = \mathcal{L}(g'(t)) = pG(p) - g(0) = pG(p),$$

where we used the Laplace's transform of the derivative and we noted by G the Laplace's transform of g, that is

$$G(p) = \mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(\tau)\mathrm{d}\tau\right),$$

such that the proof is closed. ■

Proposition 3.1.6 *Let f be an original function whose Laplace's transform is F. If we suppose that the improper integral*

$$\int_p^\infty F(q)\mathrm{d}q$$

is convergent, then

$$\int_p^\infty F(q)\mathrm{d}q = \mathcal{L}\left(\frac{f(t)}{t}\right).$$

Proof Taking into account the expression of F, we obtain

$$\int_p^\infty F(q)\mathrm{d}q = \int_p^\infty \left\{\int_0^\infty f(t)e^{-qt}\mathrm{d}t\right\}\mathrm{d}q =$$
$$= \int_0^\infty \left\{\int_p^\infty e^{-qt}\mathrm{d}q\right\} f(t)\mathrm{d}t = \int_0^\infty \left(\frac{e^{-qt}}{t}\bigg|_p^\infty\right) f(t)\mathrm{d}t =$$
$$= \int_0^\infty \frac{f(t)}{t}e^{-pt}\mathrm{d}t = \mathcal{L}\left(\frac{f(t)}{t}\right).$$

that is we have obtained the desired result. ■

Proposition 3.1.7 *If the argument of the original function f is "late", then the following formula holds*

$$\mathcal{L}(f(t-\tau)) = e^{-p\tau}F(p),\ \forall \tau > 0,$$

where, as usually, F is the Laplace's transform of the original function f.

Proof Starting from the definition of the Laplace's transform, we obtain

$$\mathcal{L}(f(t-\tau)) = \int_0^\infty f(t-\tau)e^{-pt}\mathrm{d}t,$$

such that if we use the change of variable $t-\tau=u$, it follows

$$\mathcal{L}(f(t-\tau)) = \int_{-\tau}^{\infty} f(u)e^{-pu}e^{-p\tau}\mathrm{d}u =$$
$$= \int_{-\tau}^{0} f(u)e^{-pu}e^{-p\tau}\mathrm{d}u + e^{-p\tau}\int_{0}^{\infty} f(u)e^{-pu}\mathrm{d}u =$$
$$= e^{-p\tau}\int_{0}^{\infty} f(u)e^{-pu}\mathrm{d}u = e^{-p\tau}F(p),$$

since the function f is an original and then $f(u)=0,\ \forall u<0$. ∎

It has been ascertained that although the product of two original functions is an original function, however one cannot compute the Laplace's transform for the product. But, if the usual product is substituted by the product of convolution, one can compute the Laplace's transform for this product. We know that the product of convolution for two functions can be computed in a more general context. In the case of the original functions, the product of convolution is defined by

$$(f*g)(t) = \int_{0}^{t} f(\tau)g(t-\tau)\mathrm{d}\tau. \tag{3.1.6}$$

Remark. One verifies without difficulty the following properties of the product of convolution:

– $f*g=g*f$;
– $f*(g*h)=(f*g)*h=f*g*h$;
– $f*(g+h)=f*g+f*h$;
– if $f*g=0$ then $f\equiv 0$ or $g\equiv 0$.

Proposition 3.1.8 *If f and g are original functions, then their product of convolution (3.1.6) is an original function.*

Proof The conditions (i) and (ii) from the definition of the original are immediately satisfied, taking into account that f and g satisfy these conditions. Since f and g satisfy the condition (iii), we have

$$|f(t)| \le M_1e^{s_1t},\ |g(t)| \le M_2e^{s_2t},$$

such that

$$|(f*g)(t)| \le \int_{0}^{t} |f(\tau)||g(t-\tau)|\mathrm{d}\tau \le M_1M_2\int_{0}^{t} e^{s_1\tau}e^{s_2(t-\tau)}\mathrm{d}\tau.$$

If $s_2 \le s_1$ then

$$|(f*g)(t)| \le M_1 M_2 \int_0^t e^{s_1\tau} e^{s_1(t-\tau)} \mathrm{d}\tau = M_1 M_2 \int_0^t e^{s_1 t} \mathrm{d}\tau = M_1 M_2 t e^{s_1 t}.$$

It is evident the fact that $t+1 \le e^t \Rightarrow t \le e^t - 1 \le e^t$. Then

$$|(f*g)(t)| \le M_1 M_2 e^{(s_1+1)t}.$$

If $s_1 < s_2$, then we change, reciprocally, the function f and g and use the commutativity of the product of convolution. ■

Proposition 3.1.9 *If f and g are original functions, then the Laplace's transform of their product of convolution is equal to the usual product of transforms.*

$$\mathcal{L}(f*g) = F(p).G(p).$$

Proof Taking into account Eq. (3.1.6), we obtain

$$\mathcal{L}\left((f*g)(t)\right) = \mathcal{L}\left(\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\right) =$$
$$= \int_0^\infty \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau e^{-pt}\mathrm{d}t = \int_0^\infty \int_\tau^\infty g(t-\tau)e^{-pt}\mathrm{d}t f(\tau)\mathrm{d}\tau =$$
$$= \int_0^\infty \int_0^\infty g(u)e^{-p(\tau+u)}\mathrm{d}u f(\tau)\mathrm{d}\tau = \int_0^\infty f(\tau)e^{-p\tau} \int_0^\infty g(u)e^{-pu}\mathrm{d}u\mathrm{d}\tau =$$
$$= \int_0^\infty f(\tau)e^{-p\tau}G(p)\mathrm{d}\tau = G(p)\int_0^\infty f(\tau)e^{-p\tau}\mathrm{d}\tau = G(p).F(p),$$

in which we used the change of variable $t - \tau = u$. The proposition has been proved. ■

Corollary 3.1.3 *In applications it is useful the following formula called the Duhamel's formula*

$$pF(p)G(p) = \mathcal{L}\left(f(t)g(0) + \int_0^t f(\tau)g'(t-\tau)\mathrm{d}\tau\right).$$

Proof We denote by h the product of convolution of the functions f and g, that is

$$h(t) = (f*g)(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau.$$

Then, we deduce that $h(0) = 0$ and

$$h'(t) = f(t)g(0) + \int_0^t f(\tau)g'(t-\tau)\mathrm{d}\tau.$$

Applying the Laplace's transform of the product of convolution then the Laplace's transform of the derivative and use the fact that $h(0) = 0$ and we obtain the Duhamel's formula. ■

3.2 Operational Methods

The Laplace's transform is a useful instrument to transform the mathematical operations of mathematical analysis, in more accesible operations. For instance, by using the Laplace's transform the solution of a differential equation (or of an integral equation) reduces to the solution of some algebraical equations. So, by applying the Laplace's transform a problem becomes more accessible, but the solution will be obtained in the set of the images, although the initial problem was stated in the set of the originals. Therefore, we must transpose the solution of the respective problem, from the set of the images in the set of originals. This is the subject of the so called "operational formulas", or, "operational methods".

We formulate and prove two results in this sense, which are more common.

Theorem 3.2.1 *If the series*

$$\sum_{k=1}^{\infty} \frac{c_k}{p^k} \tag{3.2.1}$$

is convergent for $|p| \geq R$, then the function

$$\theta(t) \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1}, \tag{3.2.2}$$

is an original function and its Laplace's transform is the series given in Eq. (3.2.1). Here, we have noted by θ the Heaviside's function.

Proof According to the Cauchy's criterion of convergence, we have

$$c_k \leq MR^k \Rightarrow \left| \theta(t) \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1} \right| \leq$$

$$\leq M \sum_{k=1}^{\infty} \frac{R^k |t|^{k-1}}{(k-1)!} \leq MRe^{R|t|},$$

from where we will deduce that the function (2.2) is an original function.

For the second statement of the theorem, we use the formula for the transformation of the function $f(t) = t^k$:

$$\mathcal{L}\left(\theta(t)t^{k-1}\right) = \frac{(k-1)!}{p^k}.$$

Then, based on the linearity of the Laplace's transform, we have

$$\mathcal{L}\left(\theta(t)\sum_{k=1}^{\infty}\frac{c_k}{(k-1)!}t^{k-1}\right) = \int_0^{\infty}\theta(t)\sum_{k=1}^{\infty}\frac{c_k}{(k-1)!}t^{k-1}e^{-pt}\mathrm{d}t =$$
$$= \sum_{k=1}^{\infty}\int_0^{\infty}e^{-pt}\theta(t)\frac{t^{k-1}}{(k-1)!}c_k\mathrm{d}t = \sum_{k=1}^{\infty}\frac{c_k}{p^k},$$

that is, even the desired result. ■

Theorem 3.2.2 *Let P and Q be two polynomials such that degree $P <$ degree Q and Q has only simple roots $p_0, p_1, ..., p_n$. Then the function*

$$F(p) = \frac{P(p)}{Q(p)}$$

is the Laplace's transform of the function f given by

$$f(t) = \sum_{k=0}^{n}\frac{P(p_k)}{Q'(p_k)}e^{p_k t}.$$

Proof Taking into account the hypothesis on the polynomial Q, we can write

$$Q(p) = c(p - p_0)(p - p_1)...(p - p_n)$$

and then we decompose the function F in simple fractions

$$F(p) = \frac{a_0}{p - p_0} + \frac{a_1}{p - p_1} + ... + \frac{a_n}{p - p_n}. \quad (3.2.3)$$

It is easy to see that the function F has the simple poles $p_0, p_1, ..., p_n$. Consider the circles $c_j(p_j, r_j)$ with the centers in the points p_j and the radius r_j, sufficient small such that in each closed disc does not lie any other pole, except the center of the respective circle. The coefficients a_j will be determined by integrating the equality (2.3) on the circles c_j:

$$\int_{c_j} F(p)\mathrm{d}p = \sum_{k=0}^{n} a_k \int_{c_j}\frac{1}{p - p_k}\mathrm{d}p. \quad (3.2.4)$$

According to Cauchy's theorem, the integrals from the right-hand side of the relation (3.2.4) are null, excepting the integral corresponding to $k = j$, for which we have

$$\int_{c_j} \frac{1}{p - p_j} \mathrm{d}p = 2\pi i.$$

then the relation (3.2.4) becomes

$$\int_{c_j} F(p)\mathrm{d}p = 2\pi i a_j. \tag{3.2.5}$$

On the other hand, the integral from the right-hand side of the relation (3.2.4) one can compute with the aid of the residue's theorem:

$$\int_{c_j} F(p)\mathrm{d}p = 2\pi i \ \mathrm{res}(F, p_j) = 2\pi i \frac{P(p_j)}{Q'(p_j)},$$

such that substituting in Eq. (3.2.5), we obtain

$$a_j = \frac{P(p_j)}{Q'(p_j)}.$$

Then formula (3.2.3) becomes

$$F(p) = \sum_{k=0}^{n} \frac{P(p_k)}{Q'(p_k)} \frac{1}{p - p_k} = \sum_{k=0}^{n} \frac{P(p_k)}{Q'(p_k)} \mathcal{L}\left(e^{p_k t}\right).$$

Finally, by using the linearity of the Laplace's transform, we will deduce

$$F(p) = \mathcal{L}\left(\sum_{k=0}^{n} \frac{P(p_k)}{Q'(p_k)} e^{p_k t}\right),$$

that leads to the desired result. ■

Corollary 3.2.1 *If one of the roots of the polynomial Q is null, then the original function becomes*

$$f(t) = \frac{P(0)}{Q(0)} + \sum_{k=1}^{n} \frac{P(p_k)}{R'(p_k)} e^{p_k t}, \tag{3.2.6}$$

where R is the polynomial defined such that $Q(p) = pR(p)$.

Proof We suppose that the null root is $p_0 = 0$. Then we write $Q(p) = pR(p)$. Therefore $Q'(p) = R(p) + R'(p)$. For the other roots of Q we have that $Q(p_k) = 0 \Leftrightarrow$

$R(p_k) = 0$. Then $Q'(p_k) = R(p_k) + p_k Q'(p_k) = p_k Q'(p_k)$. Therefore, the desired result follows with the aid of Theorem 1.6. ∎

Formula (3.2.6) is known as the Heaviside's formula.

In the final part of this paragraph, we want to find the image through the Laplace's transform of two functions, which is useful in many applications. Consider, firstly, the function $f(t) = t^\alpha$, where α is a complex constant such that $Re(\alpha) > -1$. If $Re(\alpha) \geq 0$, then f is an original function and then its Laplace's transform is:

$$\mathcal{L}(t^\alpha) = \int_0^\infty t^\alpha e^{-pt} \mathrm{d}t. \tag{3.2.7}$$

If $Re(\alpha) \in (-1, 0)$, then

$$\lim_{t \searrow 0} f(t) = \infty$$

and f is not an original, but the integral (3.2.7) is convergent, such that one can study the integral (3.2.7) for $Re(\alpha) > -1$. Taking into account the definition of the function Γ of Euler, from Eq. (3.2.7) we obtain that

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha + 1)}{p^{\alpha+1}}. \tag{3.2.8}$$

We must outline that formula (3.2.8) gives us the possibilty to prove, by using a new procedure, the connection between the Euler's functions, Γ and β:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \ Re(x) > -1, \ Re(y) > -1.$$

Indeed, if we begin from the equalities

$$\mathcal{L}(t^{x-1}) = \frac{\Gamma(x)}{p^x}, \ \mathcal{L}(t^{y-1}) = \frac{\Gamma(y)}{p^y},$$

and we take into account the Proposition 3.1.6, with regard to for the product of convolution, we have

$$\frac{\Gamma(x)\Gamma(y)}{p^{x+y}} = \mathcal{L}\left(t^{x+y+1} \int_0^\infty \theta^{x-1}(1-\theta)^{y-1} \mathrm{d}\theta\right),$$

in which we used the change of variable $\tau = t\theta$. The last integral is equal to $\beta(x, y)$ and

$$\mathcal{L}(t^{x+y+1}) = \frac{\Gamma(x + y)}{p^{x+y}},$$

and then

$$\Gamma(x)\Gamma(y) = \beta(x, y)\Gamma(x + y).$$

Now, we consider the Bessel's function of first species and of the order $n \in \mathbb{N}$, J_n. It is well known fact that the function J_n admits the integral representation

$$J_n(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(t \sin\theta - n\theta)} \mathrm{d}\theta.$$

J_n is a function of the class C^1 on $\mathbb{R}$ and, more,

$$|J_n(t)| \le 1,\ \forall t \in \mathbb{R},\ \forall n \in \mathbb{N}.$$

We will deduce such that J_n is an original function with the increasing index $s_0 = 0$. The image through the Laplace's transform of the function J_n is:

$$\mathcal{L}(J_n(t)) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \mathrm{d}\theta \int_0^{\infty} e^{(i \sin\theta - p)t} \mathrm{d}t.$$

If $Re(p) > s_0 = 0$, then

$$\int_0^{\infty} e^{(i \sin\theta - p)t} \mathrm{d}t = \frac{1}{p - i \sin\theta},\ \Rightarrow \mathcal{L}(J_n(t)) = \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-in\theta}}{p - i \sin\theta} \mathrm{d}\theta.$$

By using the substitution $e^{-i\theta} = z$, the integral from the right-hand side becomes a complex integral that can be computed with the aid of the residues theorem, such that, finally, we obtain

$$\mathcal{L}(J_n(t)) = \frac{1}{\sqrt{p^2 + 1}(p + \sqrt{p^2 + 1})^n}.$$

In the particular case when $n = 0$, we obtain a result very useful for applications:

$$\mathcal{L}(J_0(2\sqrt{t}) = \frac{1}{p} e^{-\frac{1}{p}}.$$

3.3 Applications

In the final part of this chapter, we study some concrete and useful applications of the Laplace's transform.

3.4 Differential Equations with Constant Coefficients

Consider the Cauchy's problem

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + ... + a_1 x^{(1)} + a_0 x = f(t),$$
$$x(0) = x_0,\ x'(0) = x_1, ..., x^{(n-1)}(0) = x_{n-1},$$

where the function $f(t)$ and the constants $a_0, a_1, ..., a_n$ and $x_0, x_1, ..., x_{n-1}$ are given.

Applying the Laplace's transform to both sides of the equation, we obtain

$$a_n \mathcal{L}\left(x^{(n)}\right) + a_{n-1} \mathcal{L}\left(x^{(n-1)}\right) + ... + a_0 \mathcal{L}(x)) = \mathcal{L}(f(t)).$$

Using the notations

$$\mathcal{L}(x(t)) = X(p),\ \mathcal{L}(f(t)) = F(p),$$

and the Laplace's transform for the derivative of an original, we obtain the relations

$$\begin{aligned}
\mathcal{L}\left(x^{(n)}\right) &= p^n X(p) - \left[p^{n-1} x_0 + p^{n-2} x_1 + ... + x_{n-1}\right], \\
\mathcal{L}\left(x^{(n-1)}\right) &= p^{n-1} X(p) - \left[p^{n-2} x_0 + p^{n-3} x_1 + ... + x_{n-2}\right], \\
&\vdots \\
\mathcal{L}(x) &= X(p)
\end{aligned}$$

Multiplying the first equation by a_n, the second by a_{n-1},..., the last by a_0 and adding the resulting relations, the following relation is obtained

$$a_n \mathcal{L}\left(x^{(n)}\right) + a_{n-1} \mathcal{L}\left(x^{(n-1)}\right) + ... + a_0 \mathcal{L}(x)) = X(p)\left[a_n p^n + a_{n-1} p^{n-1} + ... + a_0\right] -$$
$$-x_0\left[a_n p^{n-1} + a_{n-1} p^{n-2} + ... + a_1\right] - x_1\left[a_n p^{n-2} + a_{n-1} p^{n-3} + ... + a_2\right] - ...$$

that is, an equation of the form

$$F(p) = X(p)\varphi(p) - G(p)$$

with the solution

$$X(p) = \frac{X(p) + G(p)}{\varphi(p)}.$$

The solution of the initial equation becomes

$$x(t) = \mathcal{L}^{-1}(X(p)).$$

A similar procedure is used in the case of a system of differential equations. The Laplace's transform is applied for each equation of the system and an algebraical system of equations is obtained having as unknown functions the Laplace's transforms of the initial unknown functions. After we find the actual unknowns, by applying the inverse Laplace's transform we find the solutions of the initial system.

3.5 Differential Equations with Variable Coefficients

There exists some differential equations having variable coefficients which can be solved by operational methods. For instance, the differential equations for which the coefficients are polynomials in t can be approached in this manner because such an equation contains expressions of the form

$$x, tx, t^2x, ..., x', tx', t^2x', ..., x^{(n)}, tx^{(n)}, t^2x^{(n)}, ...$$

and we can use the derivative of the Laplace's transform

$$\mathcal{L}\left((-t)^n f(t)\right) = F^{(n)}(p), \text{ where } F(p) = \mathcal{L}\left(f(t)\right).$$

Example. Let us solve the equation

$$tx'' + x' + x = 0, \text{ where } x(0) = 0, \; x'(0) = 1.$$

Using the Laplace's transform of the derivative of an original and the derivative of the Laplace's transform, we obtain

$$\begin{aligned} \mathcal{L}\left(x'\right) &= pX(p) - x(0) = pX(p), \\ \mathcal{L}\left(x''\right) &= p^2X(p) - 1, \\ \mathcal{L}\left(tx''\right) = -\left[\mathcal{L}\left(x''\right)\right]' = -\left(p^2X(p) - 1\right)' &= -2pX(p) - p^2X'(p). \end{aligned}$$

Finally, it results the following equation

$$-p^2X'(p) - 2pX(p) + pX(p) + X(p) = 0 \Rightarrow p^2X'(p) + (p-1)X(p) = 0.$$

After simple calculations, we obtain

$$\frac{\mathrm{d}x}{X} = \frac{1-p}{p^2}\mathrm{d}p \Rightarrow \ln X = -\frac{1}{p} - \ln p = \ln \frac{1}{pe^{1/p}}$$

$$X(p) = \frac{1}{pe^{1/p}} = \frac{1}{p}\left(1 - \frac{1}{1!}\frac{1}{p} + \frac{1}{2!}\frac{1}{p^2} - \frac{1}{3!}\frac{1}{p^3} + ...\right).$$

This equation can be rewritten in the form

$$\mathcal{L}\left(x(t)\right) = \frac{1}{p} - \frac{1}{1!}\frac{1}{p^2} + \frac{1}{2!}\frac{1}{p^3} + ... = \mathcal{L}(1) - \frac{1}{1!}\mathcal{L}\left(t\right) + \frac{1}{2!}\mathcal{L}\left(t^2\right) - ...$$

3.6 Integral Equations

There exists some integral equations which can be solved by operational methods. For instance, let us consider the differential-integral equation

$$x'(t) = \int_0^t x(\tau)\cos(t-\tau)\mathrm{d}\tau,\ x(0) = 1.$$

Using the Laplace's transform, it results

$$pX(p) - 1 = X(p)\frac{p}{p^2+1} \Rightarrow X(p)\left(p - \frac{p}{p^2+1}\right) = 1 \Rightarrow$$

$$\Rightarrow X(p) = \frac{p}{p^2+1} = \frac{1}{p} + \frac{1}{p^3} \Rightarrow \mathcal{L}\left(x(t)\right) = \mathcal{L}(1) + \frac{1}{2}\mathcal{L}\left(t^2\right) = \mathcal{L}\left(1 + \frac{t^2}{2}\right),$$

such that the solution of the initial equation is

$$x(t) = 1 + \frac{t^2}{2}.$$

3.7 Partial Differential Equations

Consider the following mixed initial boundary value problem

$$\begin{aligned}
&a_{11}\frac{\partial^2 f}{\partial x^2} + a_{12}\frac{\partial f}{\partial x} + b_{11}\frac{\partial^2 f}{\partial t^2} + b_{12}\frac{\partial f}{\partial x} + cf = g(x,t)\\
&f(x,0) = h_1(x),\ x \in [a,b]\\
&\frac{\partial f}{\partial x}(x,0) = h_1(x)\\
&A_1\frac{\partial f}{\partial x}(0,t) + B_1\frac{\partial f}{\partial t}(0,t) + C_1 f(0,t) = k_1(t)\\
&A_2\frac{\partial f}{\partial x}(l,t) + B_2\frac{\partial f}{\partial t}(l,t) + C_2 f(l,t) = k_2(t),\ t \in [0,\infty).
\end{aligned}$$

By using the Laplace's transform, we obtain

$$\mathcal{L}\left(\frac{\partial f}{\partial t}\right) = pF(x,p) - f(x,0) = pF(x,p) - h_1(x)$$

$$\mathcal{L}\left(\frac{\partial^2 f}{\partial t^2}\right) = p^2F(x,p) - pf(x,0) - f'(x,0) =$$

$$= p^2F(x,p) - ph_1(x) - h_2(x).$$

In this way, the previous mixt problem becomes

$$a_{11}\tfrac{\mathrm{d}^2F}{\mathrm{d}x^2}(0,p) + a_{12}\tfrac{\mathrm{d}F}{\mathrm{d}x}(0,p) + b_{11}\left[p^2F(0,p) - ph_1 - h_2\right] +$$
$$+b_{12}\left[pF(0,p) - h_1\right] + cF = \Phi$$
$$A_1\tfrac{\mathrm{d}F}{\mathrm{d}x}(0,p) + B_1\left[p^2F(0,p) - h_1(0)\right] + C_1F(0,p) = K_1(p)$$
$$A_2\tfrac{\mathrm{d}F}{\mathrm{d}x}(l,p) + B_2\left[p^2F(l,p) - h_1(l)\right] + C_2F(l,p) = K_2(p),$$

that is, a mixt problem for an ordinary differential equation.

3.8 Some Improper Integrals

We consider, directly, an improper integral which is easy calculable by using the Laplace's transform.

Let us compute the following integral, well known as the Gauss's integral

$$I = \int_0^\infty e^{-x^2}\mathrm{d}x.$$

As an auxiliary instrument, consider the integral

$$J(t) = \int_0^\infty e^{-tx^2}\mathrm{d}x,$$

such that our initial integral is

$$I = J(1).$$

Using the Laplace's transform, we obtain

$$\mathcal{L}\left(J(t)\right) = \mathcal{L}\left\{\int_0^\infty e^{-tx^2}\mathrm{d}x\right\} e^{-pt}\mathrm{d}t = \int_0^\infty \left(\int_0^\infty e^{-tx^2} e^{-pt}\mathrm{d}t\right)\mathrm{d}x =$$

$$= \int_0^\infty \mathcal{L}\left(e^{-tx^2}\right)\mathrm{d}x = \int_0^\infty \frac{1}{p+x^2}\mathrm{d}x = \frac{1}{\sqrt{p}}\operatorname{arctg}\frac{x}{\sqrt{p}}\Big|_0^\infty = \frac{1}{\sqrt{p}}\frac{\pi}{2}.$$

It is known that

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{p}}$$

such that we have

$$\mathcal{L}\left(J(t)\right) = \frac{\pi}{2}\mathcal{L}\left(\frac{1}{\sqrt{\pi t}}\right) = \mathcal{L}\left(\frac{\pi}{2\sqrt{\pi t}}\right).$$

Then we deduce that

$$J(t) = \frac{\pi}{2\sqrt{\pi t}},$$

and, consequently,

$$I = J(1) = \frac{\sqrt{\pi}}{2}.$$

Chapter 4
Fourier's Transform

4.1 Fourier Series

Consider the trigonometrical series of the following form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x) . \tag{4.1.1}$$

Since the functions $\cos n\omega x$ and $\sin n\omega x$ are periodical functions having the period $T = 2\pi/\omega$ we say that the series (4.1.1) is a *periodical series*.

Let us suppose that the series (4.1.1) is convergent. Denoting by $f(x)$ its sum, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x) . \tag{4.1.2}$$

Of course, the function f is periodical with the same period, $T = 2\pi/\omega$. Also, if the coefficients of the series are changed, without loss the convergence, another sum is obtained. The reciprocal problem is of interest. If the sum $f(x)$ is fixed, how we can determine the coefficients with the help of the function $f(x)$. So, in the following we find the coefficients such that the sum of the series is the function $f(x)$.

Theorem 4.1.1 *The coefficients of the series (4.1.2) have the following expressions*

$$a_0 = \frac{2}{T}\int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x,$$

$$a_n = \frac{2}{T}\int_{\alpha}^{\alpha+T} f(x)\cos n\omega x\,\mathrm{d}x, \tag{4.1.3}$$

M. Marin and A. Öchsner, *Complements of Higher Mathematics*,
https://doi.org/10.1007/978-3-319-74684-5_4

$$b_n = \frac{2}{T}\int_{\alpha}^{\alpha+T} f(x)\sin n\omega x \mathrm{d}x.$$

Proof We start by integrating the equality (4.1.2) on an interval of length T, say $[\alpha, \alpha+T]$:

$$\int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x = \int_{\alpha}^{\alpha+T} \frac{a_0}{2}\mathrm{d}x + \sum_{n=1}^{\infty}\int_{\alpha}^{\alpha+T} (a_n\cos n\omega x + b_n \sin n\omega x)\,\mathrm{d}x \Rightarrow$$

$$\Rightarrow \int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x = \frac{a_0}{2}T + \sum_{n=1}^{\infty}\left(\frac{a_n}{n\omega}\sin n\omega x|_{\alpha}^{\alpha+T} - \frac{b_n}{n\omega}\cos n\omega x|_{\alpha}^{\alpha+T}\right).$$

Since

$$\sin n\omega(\alpha+T) - \sin n\omega\alpha = 0,\ \cos n\omega(\alpha+T) - \cos n\omega\alpha = 0,$$

we obtain

$$\int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x = \frac{a_0}{2}T \Rightarrow a_0 = \frac{2}{T}\int_{\alpha}^{\alpha+T} f(x)\mathrm{d}x.$$

Also, from the above calculations we deduce that the value of α is not important. It is important the length T of the interval, such that in the following we use the interval $[0, T]$.

Multiply the equality (4.1.2) by $\cos k\omega x$ and integrating the resulting equality on the interval $[0, T]$, we obtain

$$\int_0^T f(x)\cos k\omega x\mathrm{d}x = \int_0^T \frac{a_0}{2}\cos k\omega x\mathrm{d}x +$$

$$+\sum_{n=1}^{\infty}\int_0^T (a_n\cos n\omega x\cos k\omega x + b_n\sin n\omega x\cos k\omega x)\,\mathrm{d}x \Rightarrow$$

$$\int_0^T f(x)\cos k\omega x\mathrm{d}x = \frac{a_0}{2}\int_0^T \cos k\omega x\mathrm{d}x + \tag{4.1.4}$$

$$+\sum_{n=1}^{\infty}\left[a_n\int_0^T \cos n\omega x\cos k\omega x\mathrm{d}x + b_n\int_0^T \sin n\omega x\cos k\omega x\mathrm{d}x\right].$$

The first integral on the right-hand side becomes

$$\int_0^T \cos k\omega x \mathrm{d}x = \frac{1}{k\omega} \sin k\omega x|_0^T = 0.$$

To evaluate the last two integrals in equality (4.1.4), we must consider two cases.

1. $n \neq k$. It is easy to see that

$$\int_0^T \cos n\omega x \cos k\omega x \mathrm{d}x = \frac{1}{2} \int_0^T [\cos(n+k)\omega x + \cos(n-k)\omega x] \mathrm{d}x =$$

$$= \frac{1}{2(n+k)} \sin(n+k)\omega x|_0^T + \frac{1}{2(n-k)} \sin(n-k)\omega x|_0^T = 0.$$

Also,

$$\int_0^T \sin n\omega x \cos k\omega x \mathrm{d}x = \frac{1}{2} \int_0^T [\sin(n+k)\omega x + \sin(n-k)\omega x] \mathrm{d}x =$$

$$= -\frac{1}{2(n+k)} \cos(n+k)\omega x|_0^T - \frac{1}{2(n-k)} \sin(n-k)\omega x|_0^T = 0 =$$

$$= -\frac{1}{2(n+k)} (\cos(n+k)2\pi - 1) - \frac{1}{2(n-k)} (\cos(n-k)2\pi - 1) = 0.$$

2. $n = k$ In this case the integrals become

$$\int_0^T f(x) \cos n\omega x \mathrm{d}x =$$

$$= a_n \int_0^T \cos^2 n\omega x \mathrm{d}x + b_n \int_0^T \sin n\omega x \cos n\omega x \mathrm{d}x =$$

$$= \frac{a_n}{2} \int_0^T (1 + \cos 2n\omega x) \mathrm{d}x + \frac{b_n}{2} \int_0^T \sin 2n\omega x) \mathrm{d}x =$$

$$= \frac{a_n}{2} T + \frac{a_n}{4n\omega} \sin 2n\omega x|_0^T - \frac{b_n}{4n\omega} \cos 2n\omega x|_0^T = \frac{a_n}{2} T.$$

Therefore, we deduce

$$a_n = \frac{2}{T} \int_0^T f(x) \cos n\omega x \mathrm{d}x.$$

To obtain the coefficients b_n we multiply both sides of the equality (4.1.4) by $\sin k\omega x$. The resulting equality is integrated on the interval $[0, T]$, firstly in the case $n \neq k$, then in the case $n = k$. With the same considerations as in the case of the coefficients a_n, we finally obtain

$$b_n = \frac{2}{T}\int_0^T f(x)\sin n\omega x \mathrm{d}x,$$

such that the theorem is concluded. ■

Remarks

1. The above determined coefficients are called the *Fourier coefficients* attached to the periodical function f having the period T.

2. The Fourier coefficients are still valid even in the case the series (4.1.2) is not convergent. Indeed, in the calculations to obtain the expressions of the Fourier coefficients, the equality (4.1.2) was considered as a formal relation.

3. The Fourier coefficients received the same expressions if we take another interval, but having the same length T.

Application

Let us compute the Fourier's coefficients for the periodical function f with the period $T = 2\pi$ given by

$$f : [\pi, \pi] \to R, \ f(x) = x.$$

We have

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1, \ \alpha = -\pi \Rightarrow [\alpha, \alpha + T] = [\pi, \pi].$$

Using the formulas for the coefficients, we obtain

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\mathrm{d}x = \frac{1}{\pi}\frac{x^2}{2}|_{-\pi}^{\pi} = 0,$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x\cos nx\mathrm{d}x = \frac{1}{\pi}\left[\frac{x}{n}\sin nx|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\mathrm{d}x\right] =$$

$$= \frac{1}{\pi n^2}\cos nx|_{-\pi}^{\pi} = 0,$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin nx\mathrm{d}x = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\mathrm{d}x\right] =$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n - \frac{\pi}{n}(-1)^n + \frac{1}{n^2}\sin nx|_{-\pi}^{\pi}\right] = -2\frac{(-1)^n}{n} = \frac{2}{n}(-1)^{n+1}.$$

Then, the Fourier series of function $f(x) = x$ is

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx.$$

The conditions for the convergence of series (4.1.2) are contained in the following theorem, due to Dirichlet.

Theorem 4.1.2 *Consider a periodical function $f : R \to R$, with period T, satisfying the following conditions:*

(i) f is bounded;
(ii) f has a finite number of points of discontinuity of first order in any interval of length T;
(iii) f has a finite number of interval of monotony on any interval of length T.
Then, the series (4.1.2) is convergent at any point $x_0 \in R$, namely, to $f(x_0)$ if x_0 is a point of continuity for f and, respectively, to

$$\frac{f(x_0+0)+f(x_0-0)}{2},$$

if x_0 is a point of discontinuity of first order.

Remark. With simple words, the hope of the Fourier series is to attach to nonperiodic functions, certain periodical approximations. In the sense of this approximation it is considered the following trigonometrical polynomial

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos n\omega x + b_n \sin n\omega x),$$

that is, the partial sum of order m of the series.

To evaluate the difference between the function f and its polynomial of approximation it is computed the expression

$$E = \frac{2}{T} \int_0^T (f(x) - S_m(x))^2 \, dx,$$

from where we deduce

$$\frac{a_0^2}{2} + \sum_{n=1}^{m} \left(a_n^2 + b_n^2\right) \le \frac{2}{T} \int_0^T f(x)^2 dx,$$

called the *Bessel's inequality*.

Passing to the limit in this inequality, as $m \to \infty$ we are led to

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{2}{T} \int_0^T f(x)^2 dx,$$

called the *Parseval's identity*.

In the following we will write the Fourier series for the periodical function having the particular period 2π.

Theorem 4.1.3 *Consider a periodical function $f : [-\pi, \pi] \to R$ which has the period $T = 2\pi$. Then*

(i) if f is an even function, then its Fourier series becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\mathrm{d}x,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\mathrm{d}x.$$

(ii) if f is an odd function, then its Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\mathrm{d}x.$$

Proof (i) Since f is an even function $f(-x) = f(x)$ we can write

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x =$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos nx\mathrm{d}x + \int_0^{\pi} f(x)\cos nx\mathrm{d}x\right).$$

In the last relation, the first integral reduces to the second using the substitution $x \to -x$, taking into account that the functions f and $\cos x$ are even functions. Therefore

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\mathrm{d}x.$$

Similarly, it is easy to see that

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\mathrm{d}x =$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin nx\mathrm{d}x + \int_0^{\pi} f(x)\cos nx\mathrm{d}x\right) = 0,$$

since the function $\sin x$ is an odd function.
(ii) We can use the same procedure but taking into account that f is an odd function, that is $f(-x) = -f(x)$. So, we obtain

$$a_0 = 0,\ a_n = 0,\ b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\mathrm{d}x,$$

and the theorem is concluded. ■

Remark. It is usual to say that an even function has a Fourier's cosine series and an odd function has a Fourier's sine series.

Now, we consider the case of the functions defined on a non-symmetrical interval of the form $[0, \pi]$.

Theorem 4.1.4 *Consider the function* $f : [0, \pi] \to R$. *Then it admits both a Fourier's cosine series and a Fourier's sine series.*

Proof To find the Fourier's cosine series of the function, we construct the following function

$$g(x) = \begin{cases} f(x), & x \in [0, \pi] \\ f(-x), & x \in [-\pi, 0]. \end{cases}$$

It is a simple matter to verify that g is an even function and then, according to the Theorem 4.1.3, it admits a cosine series

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} g(x)\mathrm{d}x,\ a_n = \frac{2}{\pi}\int_0^{\pi} g(x)\cos nx\mathrm{d}x.$$

But on the interval $[0, \pi]$ the function $g(x)$ is $f(x)$ such that the above series is, in fact, the series of function f.

To find the Fourier's sine series of the function, we construct the following function

$$h(x) = \begin{cases} f(x), & x \in [0, \pi] \\ -f(-x), & x \in [-\pi, 0]. \end{cases}$$

It is a simple matter to verify that h is an odd function and then, according to the Theorem 4.1.3, it admits a sinus series

$$h(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin nx \mathrm{d}x.$$

But on the interval $[0, \pi]$ the function $h(x)$ is $f(x)$ such that the above series is, in fact, the series of function f. So, the theorem is concluded. ■

In the following we consider the general case of a function defined on an arbitrary interval $[a, b]$.

Theorem 4.1.5 *Consider the function* $f : [a, b] \to R$. *Then its Fourier series is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$a_0 = \frac{2}{b-a} \int_a^b f(x) \mathrm{d}x,$$
$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos \frac{n\pi}{b-a}(2x - a - b) \mathrm{d}x,$$
$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin \frac{n\pi}{b-a}(2x - a - b) \mathrm{d}x.$$

Proof Let $g(x)$ be the function

$$g(x) = f\left(\frac{a+b}{2} + \frac{b-a}{2\pi} x\right).$$

In order to find the domain of definition of g we observe that

$$a \leq \frac{a+b}{2} + \frac{b-a}{2\pi} x \leq b,$$

since f is defined on the interval $[a, b]$. The above inequalities become

$$2a\pi \leq a\pi + b\pi + (b-a)x \leq 2b\pi \Rightarrow x \in [-\pi, \pi].$$

Then, based on the Theorem 4.1.3 the Fourier series of g is

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\mathrm{d}x.$$

But the expression of a_n is

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{a+b}{2} + \frac{b-a}{2\pi}x\right)\cos nx\mathrm{d}x.$$

Let us make the change of variable

$$\frac{a+b}{2} + \frac{b-a}{2\pi}x = y.$$

Then

$$\mathrm{d}x = \frac{2\pi}{b-a}\mathrm{d}y.$$

Also, for $x = -\pi$ we obtain $y = a$ and for $x = \pi \Rightarrow y = b$. Then

$$x = \left(y - \frac{a+b}{2}\right)\frac{2\pi}{b-a}(2y-a-b)\frac{\pi}{b-a}.$$

Therefore

$$a_n = \frac{1}{\pi}\int_a^b f(y)\cos\frac{n\pi}{b-a}(2y-a-b)\frac{2\pi}{b-a}\mathrm{d}y =$$
$$= \frac{2}{b-a}\int_a^b f(y)\cos\frac{n\pi}{b-a}(2y-a-b)\mathrm{d}y.$$

Following the same procedure, the coefficients b_n become

$$b_n = \frac{2}{b-a}\int_a^b f(y)\sin\frac{n\pi}{b-a}(2y-a-b)\mathrm{d}y,$$

such that the theorem is concluded. ■

Remark. In the particular case $a = -l$ and $b = l$ we obtain that $f : [-l, l] \to R$ and its Fourier series has the coefficients

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) \mathrm{d}x,$$
$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x \mathrm{d}x,$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi}{l} x \mathrm{d}x.$$

In the final part of this paragraph, we give the complex form for the Fourier series. In the context of complex functions it is a very useful relation

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y),$$

called the *Euler's identity*.

Based on the Euler's identity it is easy to obtain the relations

$$\cos n\omega x = \frac{e^{in\omega x} + e^{-in\omega x}}{2}, \ \sin n\omega x = \frac{e^{in\omega x} - e^{-in\omega x}}{2i} = -i \frac{e^{in\omega x} - e^{-in\omega x}}{2}.$$

Using these relations, the Fourier series becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \frac{e^{in\omega x} + e^{-in\omega x}}{2} - i b_n \frac{e^{in\omega x} - e^{-in\omega x}}{2} \right] =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{a_n - i b_n}{2} e^{in\omega x} + \frac{a_n + i b_n}{2} e^{-in\omega x} \right).$$

Let us denote

$$c_n = \frac{a_n - i b_n}{2}.$$

Taking into account the expressions for a_n and b_n we obtain

$$c_n = \frac{1}{2} \left[\frac{2}{T} \int_0^T f(t) \cos n\omega t \mathrm{d}t - i \frac{2}{T} \int_0^T f(t) \sin n\omega t \mathrm{d}t \right] =$$

$$= \frac{1}{T} \int_0^T f(t) [\cos n\omega t - i \sin n\omega t] \mathrm{d}t = \frac{1}{T} \int_0^T f(t) e^{-in\omega t} \mathrm{d}t.$$

Similarly, we have

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{T}\int_0^T f(t)e^{in\omega t}\mathrm{d}t.$$

Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{T}\int_0^T f(t)e^{-in\omega t}\mathrm{d}t x e^{in\omega x} + \frac{2}{T}\int_0^T f(t)e^{in\omega t}\mathrm{d}t x e^{-in\omega x}\right) =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{T}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t + \frac{2}{T}\int_0^T f(t)e^{-in\omega(x-t)}\mathrm{d}t\right) =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{2}{T}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t + \sum_{n=-1}^{-\infty}\frac{2}{T}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t =$$

$$= \sum_{n=0}^{\infty}\frac{2}{T}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t + \sum_{n=-1}^{-\infty}\frac{2}{T}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t.$$

Finally, we can write

$$f(x) = \sum_{n=0}^{\infty}\frac{2}{T}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t + \sum_{n=-1}^{-\infty}\frac{2}{T}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t,$$

or,

$$f(x) = \frac{2}{T}\sum_{n=-\infty}^{\infty}\int_0^T f(t)e^{in\omega(x-t)}\mathrm{d}t.$$

4.2 Fourier's Single Integral Formula

Consider a function $f : R \to K$, where $K = R$ or $K = C$, having the following properties:

(1) f is perhaps derivable on R;

(2) In any point of discontinuity of first order t_0 f takes the value

$$f(t_0) = \frac{f(t_0 - 0) + f(t_0 + 0)}{2}.$$

(3) f is an absolutely integrable function on R, that is

$$\int_{-\infty}^{\infty} |f(t)|\mathrm{d}t < \infty.$$

Then the following formula takes place

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(\tau) e^{iu\tau} d\tau \right\} du, \tag{4.2.1}$$

which is called the *Fourier's integral formula*.

In the following theorem we pass from the complex form of the Fourier's integral formula to its real (or trigonometrical) form.

Theorem 4.2.1 *In the same conditions imposed to the function f we have the real form of the Fourier's integral formula:*

$$f(t) = \frac{1}{\pi} \int_{0}^{\infty} \left\{ \int_{-\infty}^{\infty} f(\tau) \cos u(t-\tau) d\tau \right\} du. \tag{4.2.2}$$

Proof Using the well known Euler's relation

$$e^{iu(t-\tau)} = \cos u(t-\tau) + i \sin u(t-\tau),$$

formula (4.2.1) becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(\tau) \cos u(t-\tau) d\tau + \int_{-\infty}^{\infty} f(\tau) \sin u(t-\tau) d\tau \right\} du. \tag{4.2.3}$$

Based on the notations:

$$\varphi(u,t) = \int_{-\infty}^{\infty} f(\tau) \cos u(t-\tau) d\tau,$$

$$\psi(u,t) = \int_{-\infty}^{\infty} f(\tau) \sin u(t-\tau) d\tau,$$

the relation (4.2.3) becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(u,t) du + \frac{i}{2\pi} \int_{-\infty}^{\infty} \psi(u,t) du.$$

It is easy to see that

$$\varphi(-u,t) = \varphi(u,t), \;\; \psi(-u,t) = -\psi(u,t),$$

that is, $\varphi(u,t)$ is an even function and $\psi(u,t)$ is an odd function, with regard to u.

Therefore

$$\varphi(u,t)=\int_{-\infty}^{\infty} f(\tau)\cos u(t-\tau)\mathrm{d}\tau=\int_{-\infty}^{0} f(\tau)\cos u(t-\tau)\mathrm{d}\tau+$$

$$+\int_{0}^{\infty} f(\tau)\cos u(t-\tau)\mathrm{d}\tau=2\int_{0}^{\infty} f(\tau)\cos u(t-\tau)\mathrm{d}\tau,$$

where we have used the change of variable $u\to -u$ on the interval $(\infty,0]$ and taken into account the parity of the function φ. Using the same change of variable and taking into account the parity of function ψ we obtain

$$\psi(u,t)=\int_{-\infty}^{\infty} f(\tau)\sin u(t-\tau)\mathrm{d}\tau=$$

$$=\int_{-\infty}^{0} f(\tau)\sin u(t-\tau)\mathrm{d}\tau+\int_{0}^{\infty} f(\tau)\sin u(t-\tau)\mathrm{d}\tau=0.$$

Finally, we deduce

$$f(t)=\frac{2}{\pi}\int_{-\infty}^{\infty}\varphi(u,t)\mathrm{d}u=\frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(\tau)\cos u(t-\tau)\mathrm{d}\tau\right\}\mathrm{d}u,$$

and the theorem is concluded. ∎

Remark. It is interesting to observe the analogy between the Fourier's series and the Fourier's integral formula. Indeed, observing that

$$\cos u(t-\tau)=\cos ut\cos u\tau+\sin ut\sin u\tau,$$

the real form of the Fourier's integral formula becomes

$$f(t)=\frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(\tau)\cos ut\cos u\tau\mathrm{d}\tau+\int_{-\infty}^{\infty} f(\tau)\sin ut\sin u\tau\mathrm{d}\tau\right\}\mathrm{d}u=$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\left\{\left[\int_{-\infty}^{\infty} f(\tau)\cos u\tau\mathrm{d}\tau\right]\cos ut+\left[\int_{-\infty}^{\infty} f(\tau)\sin u\tau\mathrm{d}\tau\right]\sin ut\right\}\mathrm{d}u.$$

Therefore, we can write

$$f(t)=\int_{0}^{\infty}\left[A(u)\cos ut+B(u)\sin ut\right]\mathrm{d}t, \tag{4.2.4}$$

where we have used the notations

$$A(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\cos u\tau \mathrm{d}\tau,$$

$$B(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\sin u\tau \mathrm{d}\tau.$$

Remember that the general form of the Fourier's series is

$$f(t) = \sum_{0}^{\infty} \left[a_n \cos nt + b_n \sin nt\right],$$

it is clear that this formula is analogous with Eq. (4.2.4). The sign of integral is substituted by the sign of the sum.

In the following we obtain a particular form for the Fourier's integral formula in the case of a function having parity.

Theorem 4.2.2 *Assume satisfied the standard conditions imposed to function* f. *Then*
(i) if f *is an even function then the Fourier's integral formula becomes:*

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty} \left\{\cos ut \int_{0}^{\infty} f(\tau)\cos u\tau \mathrm{d}\tau\right\} \mathrm{d}u; \tag{4.2.5}$$

(ii) if f *is an odd function then the Fourier's integral formula becomes:*

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty} \left\{\sin ut \int_{0}^{\infty} f(\tau)\sin u\tau \mathrm{d}\tau\right\} \mathrm{d}u. \tag{4.2.6}$$

Proof We write the Fourier's integral formula in the form

$$f(t) = \int_{0}^{\infty} \left[A(u)\cos ut + B(u)\sin ut\right] \mathrm{d}t, \tag{4.2.7}$$

where

$$A(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\cos u\tau \mathrm{d}\tau,$$

$$B(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\sin u\tau \mathrm{d}\tau.$$

We write

$$A(u) = \frac{1}{\pi}\left[\int_{-\infty}^{0} f(\tau)\cos u\tau \mathrm{d}\tau + \int_{0}^{\infty} f(\tau)\cos u\tau \mathrm{d}\tau\right].$$

For the first integral we make the change of variable $\tau \to -\tau$ and this integral is transformed in the second integral since the function $f(\tau)\cos u\tau$ is an even function.

Thus

$$A(u)=\frac{2}{\pi}\int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau.$$

Taking into account that the function $f(\tau)\sin u\tau$ is an even function, we obtain

$$B(u)=\frac{1}{\pi}\int_{-\infty}^\infty f(\tau)\sin u\tau \mathrm{d}\tau=$$

$$=\frac{1}{\pi}\left[\int_{-\infty}^0 f(\tau)\cos u\tau \mathrm{d}\tau+\int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau\right]=0.$$

With these evaluations about $A(u)$ and $B(u)$ the relation (4.2.7) becomes

$$f(t)=\int_0^\infty A(u)\cos ut\mathrm{d}u=\frac{2}{\pi}\int_0^\infty\left\{\cos ut\int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau\right\}\mathrm{d}u,$$

that is, the relation (4.2.5) is proved. Using similar calculations it is easy to prove the relation (4.2.6) and the theorem is concluded. ■

Application. Let us write the Fourier's integral formula for the function

$$f:R\to R,\quad f(t)=\begin{cases}1, & t\in(-a,a)\\ 1/2, & t=\pm a\\ 0, & t\in(-\infty,-a)\cup(a,\infty),\end{cases}$$

where a is a positive constant. This is the Dirichlet function of discontinuity. It is easy to see that $f(-t)=f(t)$, that is f is an even function and then

$$f(t)=\frac{2}{\pi}\int_0^\infty\left\{\cos ut\int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau\right\}\mathrm{d}u=$$

$$=\frac{2}{\pi}\int_0^\infty\left\{\cos ut\int_0^a f(\tau)\cos u\tau \mathrm{d}\tau\right\}\mathrm{d}u=\frac{2}{\pi}\int_0^\infty\left\{\cos ut\int_0^a \cos u\tau \mathrm{d}\tau\right\}\mathrm{d}u=$$

$$=\frac{2}{\pi}\int_0^\infty\left\{\cos ut\left(\frac{\sin au}{u}\right)\right\}\mathrm{d}u=\frac{2}{\pi}\int_0^\infty\frac{\cos ut\sin au}{u}\mathrm{d}u.$$

At the end of this paragraph we study, in short, the Fourier's transform, starting from the Fourier's integral formula.

Using the complex form of the Fourier's integral formula, we can write

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty\left\{\int_{-\infty}^\infty f(\tau)e^{iu\tau}\mathrm{d}\tau\right\}\mathrm{d}u=$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(\tau)e^{iut}e^{-iu\tau}\mathrm{d}\tau\right\}\mathrm{d}u =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\left\{\int_{-\infty}^{\infty} f(\tau)e^{-iu\tau}\mathrm{d}\tau\right\}\mathrm{d}u.$$

In conclusion, we have

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iut}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)e^{-iu\tau}\mathrm{d}\tau\right\}\mathrm{d}u. \tag{4.2.8}$$

By definition, the *Fourier's transform* is the function

$$\mathcal{F}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)e^{-iu\tau}\mathrm{d}\tau.$$

From Eq. (4.2.8) we immediately deduce that

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}(u)e^{iut}\mathrm{d}u,$$

which is called the *inverse Fourier's transform*.

We must outline the analogy between the Fourier's transform and its inverse, almost with regard to the kern of the transformation. Let us find the Fourier's transform in the particular case of functions having parity.

Theorem 4.2.3 *In the case f has parity, we have*

(i) if f is an even function, then its Fourier's transform becomes

$$\mathcal{F}_c(u) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\tau)\cos u\tau\mathrm{d}\tau, \tag{4.2.9}$$

and it is called the Fourier's cosine transform. Its inverse transform is

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \mathcal{F}_c(u)\cos ut\mathrm{d}u;$$

(ii) if f is an odd function, then its Fourier's transform becomes

$$\mathcal{F}_s(u) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\tau)\sin u\tau\mathrm{d}\tau, \tag{4.2.10}$$

and it is called the Fourier's sine transform. Its inverse transform is

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \mathcal{F}_s(u)\sin ut\mathrm{d}u;$$

Proof (i) Using the Fourier's integral formula for an even function, we obtain

$$f(t) = \frac{2}{\pi}\int_0^\infty \left\{\cos ut \int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau\right\} \mathrm{d}u =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos ut \left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau\right\} \mathrm{d}u.$$

Therefore,

$$\mathcal{F}_c(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau,$$

and then

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(u)\cos ut \mathrm{d}u.$$

(ii) Similarly, using the Fourier's integral formula for an odd function, we obtain

$$f(t) = \frac{2}{\pi}\int_0^\infty \left\{\sin ut \int_0^\infty f(\tau)\sin u\tau \mathrm{d}\tau\right\} \mathrm{d}u =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin ut \left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\sin u\tau \mathrm{d}\tau\right\} \mathrm{d}u.$$

Therefore,

$$\mathcal{F}_s(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\sin u\tau \mathrm{d}\tau,$$

and then

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_s(u)\sin ut \mathrm{d}u,$$

and the theorem is concluded. ∎

Application. Let us compute the Fourier's transform for the function

$$f(t) = \begin{cases} e^{-at}, & t \in [0, \infty) \\ e^{at}, & t \in (-\infty, 0), \end{cases}$$

where a is a positive constant.

It is easy to prove that the given function is an even function. Therefore, it admits a Fourier's cosine transform:

$$\mathcal{F}_c(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\cos u\tau \mathrm{d}\tau = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a\tau)}\cos u\tau \mathrm{d}\tau =$$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{1}{a} e^{-a\tau)} \cos u\tau |_0^\infty - \frac{u}{a} \int_0^\infty e^{-a\tau)} \sin u\tau \mathrm{d}\tau \right) =$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{a} - \frac{u}{a} \left(-\frac{1}{a} e^{-a\tau)} \sin u\tau |_0^\infty + \frac{u}{a} \int_0^\infty e^{-a\tau)} \cos u\tau \mathrm{d}\tau \right) \right].$$

Denoting by I the initial integral, we can write

$$I = \frac{1}{a} - \frac{u^2}{a^2} \Rightarrow I = \frac{a}{u^+ a^2}.$$

In conclusion,

$$\mathcal{F}_c(u) = \sqrt{\frac{2}{\pi}} \frac{a}{u^2 + a^2}$$

and

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{F}_c(u) \cos ut \mathrm{d}u = \frac{2a}{\pi} \int_0^\infty \frac{\cos ut}{u^2 + a^2} \mathrm{d}u.$$

Let us verify that this function is the initial function. Consider the integrals

$$I_1 = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos ut}{u^2 + a^2} \mathrm{d}u,$$

$$I_2 = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin ut}{u^2 + a^2} \mathrm{d}u.$$

Then

$$I_1 + I_2 = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{iut}}{u^2 + a^2} \mathrm{d}u = \frac{1}{2}.2\pi \text{ res}(f, ai),$$

based on the theorem of residues. The residue $\text{res}(f, ai)$ can be easily computed

$$\text{res}(f, ai) = \lim_{u \to ai} (u - ai) \frac{e^{iut}}{(u - ai)(u + ai)} = \frac{e^{-at}}{2ai}.$$

Then

$$I_1 + I_2 = \frac{1}{2}.2\pi i \frac{e^{-at}}{2ai} = \frac{\pi}{a} e^{-at}.$$

Finally, we have

$$I_1 = \frac{\pi}{2a} e^{-at} \Rightarrow f(t) = \frac{2a}{\pi} \frac{\pi}{2a} e^{-at} = e^{-at}, \ \forall t \in [0, \infty).$$

Using the parity of f we conclude that

$$f(t) = \begin{cases} e^{-at}, & t \in [0, \infty) \\ e^{at}, & t \in (-\infty, 0). \end{cases}$$

Remark. If we consider the equality

$$g(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut} \mathrm{d}t,$$

where the function g and the function f is unknown, then we say that this equality is an *integral equation of Fourier type*. Similarly, in the case of functions having parity, the equalities of the form

$$g(u) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos ut \mathrm{d}t,$$

$$g(u) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin ut \mathrm{d}t,$$

are, also, integral equations of Fourier type. In each case the solutions are determined with the help of the inverse Fourier's transform.

Application. Find the solution of the integral equation

$$\int_{0}^{\infty} f(t) \cos ut \mathrm{d}t = \varphi(u),$$

where f is the unknown function and φ is given by

$$\varphi(u) = \begin{cases} 1 - u, & u \in [0, 1] \\ 0, & u \in (1, \infty). \end{cases}$$

Multiplying by $\sqrt{2/\pi}$ both sides of the equation and denoting

$$g(u) = \sqrt{\frac{2}{\pi}} \varphi(u),$$

we obtain an equation of the above form. With the help of the inverse Fourier's transform we find

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(u) \cos ut \mathrm{d}u = \frac{2}{\pi} \int_{0}^{\infty} \varphi(u) \cos ut \mathrm{d}u =$$

$$= \frac{2}{\pi} \int_{0}^{1} (1 - u) \cos ut \mathrm{d}u = \frac{2}{\pi} \frac{1 - \cos t}{t^2}, \ t \in [0, \infty).$$

4.3 Fourier's Transform in L^1

We remember, firstly, the fact that a function $f : \mathbb{R} \to \mathbb{R}$ belonging to $L^1(\mathbb{R})$ and we write shortly $f \in L^1$, if

$$\int_{-\infty}^{+\infty} |f(t)|\mathrm{d}t < +\infty.$$

Definition 4.3.1 If the function $f : R \to R$, $f \in L^1$ then its Fourier's transform is defined by

$$\mathcal{F}(f(t))(x) = \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t, \tag{4.3.1}$$

where i is the complex unit, $i^2 = -1$.

For simplicity, we use the notation $\mathcal{F}(f(t))(x) = \widehat{f}(x)$.

Theorem 4.3.1 *If $f \in L^1$, then its Fourier's transform $\widehat{f}$ is a bounded and continuous function on R. Moreover, we have*

$$\left|\widehat{f}(x)\right| \leq \|\widehat{f}\|_{B(R)} \leq \|f\|_{L^1(R)}, \tag{4.3.2}$$

where we denoted by $B(\mathbb{R})$ the set of bounded functions R.

Proof Starting from the definition (4.3.1), we obtain

$$\left|\widehat{f}(x)\right| \leq \int_{-\infty}^{+\infty} |f(t)| \left|e^{ixt}\right| \mathrm{d}t =$$
$$= \int_{-\infty}^{+\infty} |f(t)|\mathrm{d}t = \|f\|_{L^1(\mathbb{R})}.$$

If in this inequality we pass to the supremum, it results

$$\|\widehat{f}\|_{B(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})},$$

and this proves that the Fourier's transform is a bounded function. Moreover, we already proved the double inequality (4.3.2). Let us prove now that $\widehat{f}$ is a continuous function. We use the estimations:

$$\left|\widehat{f}(x+h)-\widehat{f}(x)\right| = \left|\int_{-\infty}^{+\infty} f(t)\left[e^{i(x+h)t}-e^{ixt}\right]\mathrm{d}t\right| \leq$$

$$\leq \int_{-\infty}^{+\infty} |f(t)|\left|e^{ixt}\right|\left|e^{iht}-1\right|\mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\left|e^{iht}-1\right|\mathrm{d}t \leq 2\int_{-\infty}^{+\infty} |f(t)|\mathrm{d}t \tag{4.3.3}$$

from where we will deduce that the difference from the left-hand side of the inequality (4.3.3) is bounded by a summable function.

On the other hand,

$$\left|\widehat{f}(x+h)-\widehat{f}(x)\right| \leq \int_{-\infty}^{+\infty} \left|e^{iht}-1\right| |f(t)|\mathrm{d}t$$

and

$$\lim_{h\to 0}\left|e^{iht}-1\right| |f(t)| = 0.$$

This means that the conditions of the Lebesque's theorem are valid and we can pass to the limit under the integral:

$$\lim_{h\to 0}\left|\widehat{f}(x+h)-\widehat{f}(x)\right| = \int_{-\infty}^{+\infty} \lim_{h\to 0}\left|e^{iht}-1\right| |f(t)|\mathrm{d}t = 0,$$

that is $\widehat{f}(x)$ is continuous in any point $x \in \mathbb{R}$. ■

Corollary 4.3.1 *If we have a sequence $\{f_n\}_{n\geq 1}$ of functions from $L^1(\mathbb{R})$ such that*

$$\lim_{n\to\infty} f_n = f, \text{ in } L^1(\mathbb{R}),$$

then

$$\lim_{n\to\infty} \widehat{f_n}(x) = \widehat{f}(x), \text{ uniform with regard to } x \in \mathbb{R}.$$

Proof The result immediately follows based on the inequality (4.3.2):

$$\left|\widehat{f_n}(x)-\widehat{f}(x)\right| \leq \|f_n - f\|_{L^1(\mathbb{R})},$$

from where it results the conclusion of the corollary. ■

In the following theorem we will prove the main properties of the Fourier's transform.

Theorem 4.3.2 *If $f \in L^1$, then its Fourier's transform $\widehat{f}$ satisfies the following rules of calculus*

$$\mathcal{F}(f(t+a)) = e^{-iax}\mathcal{F}(f(t)) = e^{-iax}\widehat{f}(x),$$
$$\widehat{f}(x+b) = \mathcal{F}\left(e^{ibt}f(t)\right) = \widehat{e^{ibt}f(t)}. \tag{4.3.4}$$

Proof We begin from the definition of the Fourier's transform:

$$\mathcal{F}(f(t+a)) = \int_{-\infty}^{+\infty} f(t+a)e^{ixt}\mathrm{d}t =$$
$$= e^{-iax}\int_{-\infty}^{+\infty} f(\tau)e^{ix\tau}\mathrm{d}\tau = e^{-iax}\widehat{f}(x),$$

where we used the change of variable $t+a=\tau$. Thus, we already proved the formula $(4.3.4)_1$. In view of formula $(4.3.4)_2$ we begin also from the definition of the Fourier's transform

$$\mathcal{F}(e^{ibt}f(t)) = \widehat{e^{ibt}f(t)} = \int_{-\infty}^{+\infty} e^{ibt}f(t)e^{ixt}\mathrm{d}t =$$
$$= \int_{-\infty}^{+\infty} f(t)e^{i(x+b)t}\mathrm{d}t = \widehat{f}(x+b),$$

that is we have obtained $(4.3.4)_2$. ■

In the following theorem, due to the great mathematicians Riemann and Lebesque, we study the behavior of the Fourier's transform to the infinity.

Theorem 4.3.3 *If $f \in L^1$, then*

$$\lim_{x\to\pm\infty}\widehat{f}(x) = \lim_{x\to\pm\infty}\int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t = 0.$$

Proof We can write

$$-\widehat{f}(x) = e^{i\pi}\widehat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{i\pi}e^{ixt}\mathrm{d}t =$$
$$\int_{-\infty}^{+\infty} f(t)e^{ix(t+\pi/x)}\mathrm{d}t = \int_{-\infty}^{+\infty} f(\tau-\pi/x)e^{ix\tau}\mathrm{d}\tau,$$

after that we made the change of variable $t+\pi/x=\tau$.

Then

$$2\widehat{f}(x) = \widehat{f}(x) - (-\widehat{f}(x)) = \int_{-\infty}^{+\infty}\left[f(t) - f\left(t-\frac{\pi}{x}\right)\right]e^{ixt}\mathrm{d}t. \tag{4.3.5}$$

For the last integrant from (4.3.5) we have the estimation

$$\left|\left[f(t)-f\left(t-\frac{\pi}{x}\right)\right]e^{ixt}\right| \leq |f(t)|+\left|f\left(t-\frac{\pi}{x}\right)\right|,$$

that is the last integrant from Eq. (4.3.5) is bounded by a summable function (by hypothesis, $f \in L^1$). We can then use the Lebesque's theorem to pass to the limit under the integral in Eq. (4.3.5). Taking into account that

$$\lim_{x\to\pm\infty}\left|f(t)-f\left(t-\frac{\pi}{x}\right)\right|=0,$$

then immediately follows the desired result. ■

Corollary 4.3.2 *If $f \in L^1(\mathbb{R})$, then*

$$\lim_{x\to\pm\infty}\int_{-\infty}^{+\infty} f(t)\cos xt\mathrm{d}t=0,\ \lim_{x\to\pm\infty}\int_{-\infty}^{+\infty} f(t)\sin xt\mathrm{d}t=0.$$

Proof The result immediately follows from the Theorem 4.3.3, by using the Euler's formula $e^{ixt} = \cos xt + i\sin xt$. ■

According to the Theorems 4.3.2 and 4.3.3, the Fourier's transform is a continuous function on R and has null limits to $-\infty$ and to $+\infty$. Now, we consider the inverse problem. If we have a function g which is continuous on $\mathbb{R}$ and has null limits to $-\infty$ and $+\infty$, then g is the Fourier's transform of a functions from $L^1(\mathbb{R})$? The answer is negative and we prove this by using a counterexample.

Lemma 4.3.1 *If a function g has the properties of a Fourier's transform, then it is not necessary that g is the image of a function from $L^1(\mathbb{R})$.*

Proof We define the function g by

$$g(x)=\begin{cases}-g(-x)\ , \text{ if } x<0,\\ \quad x/e \quad\ , \text{ if } 0\leq x\leq e,\\ \ 1/\ln x\ \ , \text{ if } x>e.\end{cases}$$

From the definition, g is symmetric with regard to the origin. Then

$$\lim_{x\to\infty} g(x)=\lim_{x\to\infty}\frac{1}{\ln x}=0.$$

On the other hand, g is continuous, because for $x=e$, we have

$$g(e-0)=g(e+0)=1.$$

Therefore, the function g has the properties of a Fourier's transform. However, g is not the image of a function from $L^1(R)$. We suppose, through absurdum, that there exists a function $f \in L^1(R)$ such that

$$g(x) = \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t. \tag{4.3.6}$$

Let us compute the following limit:

$$\lim_{n\to\infty}\int_e^n \frac{g(x)}{x}\mathrm{d}x = \lim_{n\to\infty}\int_e^n \frac{1}{x\ln x}\mathrm{d}x =$$

$$= \lim_{n\to\infty}[\ln(\ln x)]_e^n = \lim_{n\to\infty}\ln(\ln n) = \infty. \tag{4.3.7}$$

Therefore, if we begin from the definition of g, we obtain that the limit from (4.3.7) is infinite. Let us prove that if we use the form (4.3.6) of the function g, then the limit from (4.3.7) is finite. Indeed, taking into account the form (4.3.6) of the function g, we obtain

$$g(x) = -g(-x) = \int_{-\infty}^{+\infty} f(t)e^{-ixt}\mathrm{d}t.$$

Summing this relation with (4.3.6) we are led to

$$2g(x) = \int_{-\infty}^{+\infty} f(t)\left[e^{ixt} + e^{-ixt}\right]\mathrm{d}t = 2i\int_{-\infty}^{+\infty} f(t)\sin xt\mathrm{d}t.$$

Then

$$g(x) = i\int_{-\infty}^{0} f(t)\sin xt\mathrm{d}t + i\int_0^{\infty} f(t)\sin xt\mathrm{d}t =$$
$$= i\int_0^{\infty}[f(t) - f(-t)]\sin xt\mathrm{d}t.$$

Therefore, the integral, under the limit from Eq. (4.3.7), becomes

$$\int_e^n \frac{g(x)}{x}\mathrm{d}x = i\int_e^n\left\{\int_0^{\infty}[f(t) - f(-t)]\frac{\sin xt}{x}\mathrm{d}t\right\}\mathrm{d}x. \tag{4.3.8}$$

In the last integral we can change the order of integration, because $f(t) - f(-t)$ is summable (by hypothesis, $f \in L^1(R)$). Therefore,

$$\int_e^n \frac{g(x)}{x}\mathrm{d}x = i\int_0^{\infty}[f(t) - f(-t)]\left\{\int_e^n \frac{\sin xt}{x}\mathrm{d}x\right\}\mathrm{d}t =$$
$$= i\int_0^{\infty}[f(t) - f(-t)]\left\{\int_{et}^{nt}\frac{\sin\xi}{\xi}\mathrm{d}\xi\right\}\mathrm{d}t < \infty,$$

since the integral

$$\int_{et}^{nt} \frac{\sin \xi}{\xi} \mathrm{d}\xi$$

is convergent and the function $f(t) - f(-t)$ is summable.

So, we arrive at a contradiction which proves that the function g cannot be the Fourier's transform of a function from $L^1(R)$. ■

Another natural question with regard to the Fourier's transform is the following:

If $f \in L^1(R)$ then $\widehat{f} \in L^1(R)$? The answer is again, negative and we prove this by using a counter-example.

Lemma 4.3.2 *If a function is from $L^1(\mathbb{R})$ then its Fourier's transform is not necessary to be a function from $L^1(\mathbb{R})$.*

Proof We define the function f by

$$f(t) = \begin{cases} 0 & , \text{ if } t < 0, \\ e^{-t} & , \text{ if } t \leq 0. \end{cases}$$

Since

$$\int_{-\infty}^{+\infty} f(t)\mathrm{d}t = \int_0^{+\infty} e^{-t}\mathrm{d}t = 1,$$

we will deduce that $f \in L^1(\mathbb{R})$.

But the Fourier's transform of the function f is

$$\widehat{f}(x) = \int_0^{+\infty} e^{-t}e^{ixt}\mathrm{d}t = \int_0^{+\infty} e^{(ix-1)t}\mathrm{d}t = \frac{1}{1-ix} = \frac{1+ix}{1+x^2},$$

from where, clearly, it follows that $\widehat{f} \notin L^1(\mathbb{R})$. ■

Now, we expose, without proof, two theorems, due to Jordan, which give the connection between the Fourier's transform and the original function.

Theorem 4.3.4 *If $f \in L^1(\mathbb{R})$ and, more, f is a function with bounded variation ($f \in BV(R)$), then in the close vicinity of a fixed point u it holds the following formula of inversion*

$$\lim_{a\to\infty} \frac{1}{2\pi} \int_{-a}^{a} \widehat{f}(x)e^{-ixu}\mathrm{d}x = \frac{1}{2}[f(u+0) - f(u+0)].$$

If u is a point of continuity for the function f, then the formula of inversion becomes

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(x)e^{-ixu}\mathrm{d}x,$$

where $\widehat{f}$ is the Fourier's transform of the function f.

Theorem 4.3.5 *If $f \in L^1(R)$ and $\widehat{f} \in L^1(R)$, then in a point u of continuity of the function f we have the following formula of inversion*

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(x) e^{-ixu} \mathrm{d}x.$$

At the end of this paragraph we will provide some considerations on the product of convolution for the functions from $L^1(R)$.

By definition, if $f, g \in L^1(R)$, then its product of convolution is

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-\tau) g(\tau) \mathrm{d}\tau. \tag{4.3.9}$$

Theorem 4.3.6 *If $f, g \in L^1(\mathbb{R})$ then its product of convolution is defined piecewise on $\mathbb{R}$ and is a function from $L^1(\mathbb{R})$.*

Proof By using the change of variable $t - \tau = u$, we have

$$\int_{-\infty}^{+\infty} |f(t-\tau)| \mathrm{d}\tau = \int_{-\infty}^{+\infty} |f(u)| \mathrm{d}u,$$

such that, taking into account that $f \in L^1(\mathbb{R})$, we will deduce that the integrant from the right-hand side of the relation (4.3.9) is a function defined piecewise and summable. We can, therefore, invert the order of integration:

$$\begin{aligned}\int_{-\infty}^{+\infty} |f * g|(t) \mathrm{d}t &= \textstyle\int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} |f(t-\tau)||g(\tau)| \mathrm{d}\tau \right\} \mathrm{d}t = \\ &= \textstyle\int_{-\infty}^{+\infty} |g(\tau)| \left\{ \int_{-\infty}^{+\infty} |f(t-\tau)| \mathrm{d}t \right\} \mathrm{d}\tau = \\ &= \|f\|_{L^1} \textstyle\int_{-\infty}^{+\infty} |g(\tau)| \mathrm{d}\tau = \|f\|_{L^1} \|g\|_{L^1},\end{aligned}$$

and this proves that $f * g \in L^1(\mathbb{R})$. ■

Proposition 4.3.1 *If $f, g \in L^1(\mathbb{R})$ then*

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}.$$

Proof Since $f, g \in L^1(\mathbb{R})$, then according to the Theorem 4.3.6, we have that $f * g \in L^1(\mathbb{R})$. By using the definition (4.3.9) of the product of convolution and the definition of the norm in $L^1(\mathbb{R})$, we obtain

$$\begin{aligned}
\|f * g|_{L^1} &= \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\mathrm{d}\tau \right| \mathrm{d}t \leq \\
&\leq \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} |f(t-\tau)g(\tau)|\mathrm{d}\tau \right| \mathrm{d}t = \\
&= \int_{-\infty}^{+\infty} |g(\tau)| \left\{ \int_{-\infty}^{+\infty} |f(t-\tau)|\mathrm{d}t \right\} \mathrm{d}\tau = \\
&= \int_{-\infty}^{+\infty} |g(\tau)| \left\{ \int_{-\infty}^{+\infty} |f(u)|\mathrm{d}u \right\} \mathrm{d}\tau = \\
&= \|f\|_{L^1} \int_{-\infty}^{+\infty} |g(\tau)|\mathrm{d}\tau = \|f\|_{L^1} \|g\|_{L^1},
\end{aligned}$$

such that the proof is concluded. ∎

Since we already proved that the product of convolution $f * g$ is a function from $L^1(R)$, we can compute its Fourier's transform.

Theorem 4.3.7 *If* $f, g \in L^1(R)$, *then*

$$\mathcal{F}((f * g)(t)) = \mathcal{F}(f(t)).\mathcal{F}(g(t)).$$

Proof We take into account the definition of the Fourier's transform for the functions from $L^1(R)$ and the definition of the product of convolution. So, we obtain

$$\begin{aligned}
\mathcal{F}((f * g)(t)) &= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\mathrm{d}\tau \right\} e^{ixt}\mathrm{d}t = \\
&= \int_{-\infty}^{+\infty} g(\tau) \left\{ \int_{-\infty}^{+\infty} f(t-\tau)e^{ixt}\mathrm{d}t \right\} \mathrm{d}\tau = \\
&= \int_{-\infty}^{+\infty} g(\tau) \left\{ \int_{-\infty}^{+\infty} f(u)e^{ixu}\mathrm{d}u \right\} e^{ix\tau}\mathrm{d}\tau = \\
&= \widehat{f}(x) \int_{-\infty}^{+\infty} g(\tau)e^{ix\tau}\mathrm{d}\tau = \widehat{f}(x).\widehat{g}(x),
\end{aligned}$$

after that we made the change of variable $t - \tau = u$. ∎

4.4 Fourier's Transform in L^2

The result proved in the following lemma is very useful in this paragraph.

Lemma 4.4.1 *For* $\forall \varepsilon > 0$ *and* $\forall \alpha \in R$, *we have the following equality*

$$\int_{-\infty}^{+\infty} e^{i\alpha t} e^{-\varepsilon t^2} \mathrm{d}t = \left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{\frac{-\alpha^2}{4\varepsilon}}.$$

Proof Using the change of variable

$$t = \frac{x}{\sqrt{\varepsilon}},$$

we obtain

$$\int_{-\infty}^{+\infty} e^{i\alpha t} e^{-\varepsilon t^2} \mathrm{d}t = \frac{1}{\sqrt{\varepsilon}} \int_{-\infty}^{+\infty} e^{i\alpha \frac{x}{\sqrt{\varepsilon}}} e^{-x^2} \mathrm{d}x. \tag{4.4.1}$$

We consider as well known result the value of the Gauss's integral

$$\int_{-\infty}^{+\infty} e^{-(x+i\beta)^2} \mathrm{d}x = \sqrt{\pi}. \tag{4.4.2}$$

This result can be obtained by using the Laplace's transform, or with the aid of some procedures in the context of theory of the complex integral.

We can write the integral from Eq. (4.4.2) in the form

$$\int_{-\infty}^{+\infty} e^{-x^2} e^{-2\beta x i} e^{\beta^2} \mathrm{d}x = e^{\beta^2} \int_{-\infty}^{+\infty} e^{-x^2} e^{-2\beta x i} \mathrm{d}x$$

and then

$$\int_{-\infty}^{+\infty} e^{-x^2} e^{-2\beta x i} \mathrm{d}x = \sqrt{\pi} e^{-\beta^2}.$$

Introducing this result in Eq. (4.3.1) by taking

$$\beta = -\frac{\alpha}{2\sqrt{\varepsilon}}:$$

$$\int_{-\infty}^{+\infty} e^{i\alpha t} e^{-\varepsilon t^2} \mathrm{d}t = \frac{1}{\sqrt{\varepsilon}} \sqrt{\pi} e^{\frac{-\alpha^2}{4\varepsilon}},$$

and this concludes the proof. ■

In the following theorem we prove a fundamental result, which anticipates the Fourier's transform for the functions from $L^2(R)$.

Theorem 4.4.1 *Consider the function* $f \in L^1(R) \cap L^2(R)$. *Then* $\widehat{f}$, *like a Fourier's transform of a functions from* $L^1(R)$, *is a function from* $L^2(R)$. *Moreover,*

$$\|\widehat{f}\|_{L^2(R)} = \sqrt{2\pi}\|f\|_{L^2(R)}.$$

Proof Since $f \in L^1(R)$, we know that its Fourier's transform exists, $\widehat{f}$, given by

$$\widehat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t.$$

Then

$$\left|\widehat{f}(x)\right|^2 = \widehat{f}(x)\overline{\widehat{f}(x)} = \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t \int_{-\infty}^{+\infty} \overline{f(u)}e^{-ixu}\mathrm{d}u.$$

Multiplying this equality by

$$e^{-x^2/n}$$

and then integrate the resulting equality over R:

$$I \equiv \int_{-\infty}^{+\infty} \left|\widehat{f}(x)\right|^2 e^{-\frac{x^2}{n}}\mathrm{d}x = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{n}} \left\{\int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t \int_{-\infty}^{+\infty} \overline{f(u)}e^{-ixu}\mathrm{d}u\right\}\mathrm{d}x.$$

Since f and $\overline{f}$ are absolute integrable functions, we can invert the order of integration:

$$I = \int_{-\infty}^{+\infty} \overline{f(u)} \left\{\int_{-\infty}^{+\infty} f(t)\left[\int_{-\infty}^{+\infty} e^{-\frac{x^2}{n}} e^{ix(t-u)}\mathrm{d}x\right]\mathrm{d}t\right\}\mathrm{d}u =$$
$$= \sqrt{\pi n}\int_{-\infty}^{+\infty} \overline{f(u)} \left\{\int_{-\infty}^{+\infty} f(t)e^{-\frac{n(t-u)^2}{4}}\mathrm{d}t\right\}\mathrm{d}u,$$

in which we used the result from the Lemma 4.4.1 with $\varepsilon = 1/n$ and $\alpha = t - u$.

In the last integral we make the change of variable $t - u = s$ and then we denote again s by t:

$$I = \sqrt{\pi n}\int_{-\infty}^{+\infty} \overline{f(u)} \left\{\int_{-\infty}^{+\infty} f(t+u)e^{-\frac{nt^2}{4}}\mathrm{d}t\right\}\mathrm{d}u =$$
$$= \sqrt{\pi n}\int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} \overline{f(u)}f(t+u)\mathrm{d}u\right\} e^{-\frac{nt^2}{4}}\mathrm{d}t. \tag{4.4.3}$$

We introduce the notation

$$g(t) = \int_{-\infty}^{+\infty} \overline{f(u)}f(t+u)\mathrm{d}u. \tag{4.4.4}$$

Let us prove that the function g is continuous in $t = 0$. Indeed,

$$\begin{aligned}|g(t)-g(0)|^2 &= \left|\int_{-\infty}^{+\infty}\overline{f(u)}[f(t+u)-f(u)]\mathrm{d}u\right|^2 \le \\ &\le \int_{-\infty}^{+\infty}\left|\overline{f(u)}\right|.\int_{-\infty}^{+\infty}|f(t+u)-f(u)|^2\,\mathrm{d}u = \\ &= \|f\|_{L^2}.\int_{-\infty}^{+\infty}|f(t+u)-f(u)|^2\,\mathrm{d}u,\end{aligned}$$

in which we used the inequality of Hőlder.

It is known that any function from $L^p,\ \ p>1$ (in the our case, $f\in L^2$) is continuous in mean and then

$$\int_{-\infty}^{+\infty}|f(t+u)-f(u)|^2\,\mathrm{d}u\to 0,\ \text{for } t\to 0,$$

that proves that

$$|g(t)-g(0)|^2\to 0,\ \text{for } t\to 0,$$

that is, the function g is continuous in the origin.

We come back to the relation (4.4.3) and we write it in the form

$$\begin{aligned}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{n}}\left|\widehat{f}(x)\right|^2\mathrm{d}x &= \sqrt{\pi n}\int_{-\infty}^{+\infty}e^{-\frac{nt^2}{4}}g(t)\mathrm{d}t = \\ &= 2\sqrt{\pi}\int_{-\infty}^{+\infty}e^{-\tau^2}g\left(\frac{2}{\sqrt{n}}\tau\right)\mathrm{d}\tau.\end{aligned} \tag{4.4.5}$$

From the definition (4.4.4) of the function g, we will deduce

$$\begin{aligned}|g(t)|\le\left\{\int_{-\infty}^{+\infty}\left|\overline{f}(u)\right|^2\mathrm{d}u\right\}^{1/2}.\left\{\int_{-\infty}^{+\infty}|f(t+u)|^2\,\mathrm{d}u\right\}^{1/2} &= \\ = \left(\|f\|_{L^2}^2\right)^{1/2}\left(\|f\|_{L^2}^2\right)^{1/2} = \|f\|_{L^2}^2.\end{aligned}$$

Also, from Eq. (4.4.4) we obtain

$$g(0)=\int_{-\infty}^{+\infty}\overline{f}(u)f(u)\mathrm{d}u=\int_{-\infty}^{+\infty}|f(u)|^2\,\mathrm{d}u=\|f\|_{L^2}^2. \tag{4.4.6}$$

Since the function

$$e^{-\tau^2}g\left(\frac{2}{\sqrt{n}}\tau\right)$$

is superior bounded by a summable function, namely by

$$e^{-\tau^2}\|f\|_{L^2}^2,$$

we will deduce that in Eq. (4.4.5) we can use the Lebesque's theorem which permits to pass to the limit under the integral. Thus, for $n \to \infty$, from Eq. (4.4.5) it results

$$\int_{-\infty}^{+\infty} \left|\widehat{f}(x)\right|^2 \mathrm{d}x = 2\sqrt{\pi}\int_{-\infty}^{+\infty} e^{-t^2} g(0)\mathrm{d}t =$$

$$= 2\sqrt{\pi}\|f\|_{L^2}^2 \int_{-\infty}^{+\infty} e^{-t^2}\mathrm{d}t = 2\sqrt{\pi}\|f\|_{L^2}^2\sqrt{\pi} = 2\pi\|f\|_{L^2}^2,$$

in which we used the relation (4.4.6).

Therefore,

$$\|\widehat{f}\|_{L^2}^2 = 2\pi\|f\|_{L^2}^2 \Rightarrow \|\widehat{f}\|_{L^2} = \sqrt{2\pi}\|f\|_{L^2},$$

that concludes the proof of the theorem. ■

We make now another step in view of introducing the Fourier's transform for the functions from $L^2(R)$. To this, we introduce the *truncated function*. Thus, if $f \in L^2(R)$, then the truncated function f_a, associated to f, is defined by

$$f_a(t) = \begin{cases} f(t)\ , & \text{if } |t| \le a, \\ 0\ \ , & \text{if } |t| > a. \end{cases} \tag{4.4.7}$$

Theorem 4.4.2 *If the function $f \in L^2(R)$ then the truncated function f_a is a function from $f \in L^1(R) \cap L^2(R)$ and, therefore, it admits Fourier's transform and $\widehat{f_a} \in L^2(R)$. Moreover, for $a \to 0$ we have*

$$\widehat{f_a}(t) \to \widehat{f}(t), \text{ in norm of } L^2.$$

Proof Firstly, we can observe that from Eq. (4.4.7) it results

$$|f_a(t)| \le |f(t)|,\ \forall t \in R \Rightarrow |f_a(t)|^2 \le |f(t)|^2,\ \forall t \in R.$$

Thus, by integrating the last inequality, it results

$$\int_{-\infty}^{+\infty} |f_a(t)|^2\,\mathrm{d}t \le \int_{-\infty}^{+\infty} |f(t)|^2\mathrm{d}t \Rightarrow$$

$$\Rightarrow \|f_a(t)\|_{L^2} \le \|f(t)\|_{L^2},$$

that proves that $f_a \in L^2(R)$.

On the other hand, from the definition of a truncated function we obtain

$$\int_{-\infty}^{+\infty} |f_a(t)|\,\mathrm{d}t = \int_{-a}^{+a} |f(t)|\mathrm{d}t \le \sqrt{2a}\left(\int_{-a}^{+a} |f(t)|^2\mathrm{d}t\right)^{1/2},$$

based on the inequality of Hőlder.

But

$$\int_{-a}^{+a} |f(t)|^2\mathrm{d}t \le \int_{-\infty}^{+\infty} |f(t)|^2\mathrm{d}t = \|f\|_{L^2}^2$$

and then

$$\int_{-\infty}^{+\infty} |f_a(t)|\,\mathrm{d}t \le \sqrt{2a}\|f\|_{L^2}^2$$

that proves that $f_a \in L^1(R)$. Therefore, f_a is a function from $L^1(R)$ and from $L^2(R)$ too, that is, it satisfies the hypotheses of the Theorem 4.4.1 and then there exists its Fourier's transform in the sense of transform for the functions from $L^1(R)$:

$$\widehat{f_a}(x) = \int_{-\infty}^{+\infty} f_a(t)e^{ixt}\mathrm{d}t = \int_{-a}^{+a} f(t)e^{ixt}\mathrm{d}t.$$

Also, from the Theorem 4.4.1 we will deduce that $\widehat{f_a} \in L^1(R)$ too. We must prove that $\widehat{f_a}$ is convergent in the space $L^2(R)$. To this we use the Cauchy's criterion of fundamental sequence ($L^2(R)$ is a completed space). For $b > 0$, we have

$$\left\|\widehat{f_a} - \widehat{f_{a+b}}\right\|_{L^2}^2 \le \left|\int_{-a+b}^{+a} |f(t)|^2\mathrm{d}t + \int_{a}^{a+b} |f(t)|^2\mathrm{d}t\right|.$$

Therefore, $\forall\varepsilon > 0,\ \exists n_0(\varepsilon)$ such that if $a > n_0(\varepsilon)$ and $b > 0$, we have

$$\left\|\widehat{f_a} - \widehat{f_{a+b}}\right\|_{L^2}^2 < \varepsilon$$

and this proves that the sequence $\{\widehat{f_a}\}$ is convergent in the norm of $L^2(R)$. ■

Now, we can define the Fourier's transform for a function from $L^2(R)$.

Definition 4.4.1 If the function $f \in L^2(R)$ then one can attach the truncated function f_a and to this, as a function from $L^1(R)$, one can attach the Fourier's transform

$$\widehat{f_a}(x) = \int_{-\infty}^{+\infty} f_a(t)e^{ixt}\mathrm{d}t = \int_{-a}^{+a} f(t)e^{ixt}\mathrm{d}t.$$

By definition

$$\widehat{f}(x) = \lim_{a\to 0} \widehat{f_a}(x),\ \text{ in } L^2.$$

The result from the following theorem is due to Parseval.

Theorem 4.4.3 *If the function $f \in L^2(R)$ then $\widehat{f} \in L^2(R)$ and*

$$\left\| \widehat{f} \right\|_{L^2} = \sqrt{2\pi} \, \| f \|_{L^2} .$$

Proof The fact that $\widehat{f} \in L^2(R)$ follows from Theorem 4.4.1. Then we have the inequality

$$\left| \left\| \widehat{f_n} \right\|_{L^2} - \left\| \widehat{f_m} \right\|_{L^2} \right| \le \left\| \widehat{f_n} - \widehat{f_m} \right\|_{L^2} . \tag{4.4.8}$$

In the Theorem 4.4.2 we already proved that the sequence $\{\widehat{f_n}\}$ is convergent and then

$$\lim_{n\to\infty} \left\| \widehat{f_n} \right\|_{L^2} = \left\| \widehat{f} \right\|_{L^2} .$$

On the other hand, because $f_n \in L^1(R) \cap L^2(R)$, for the truncated function we can write

$$\left\| \widehat{f_n} \right\|_{L^2} = \sqrt{2\pi} \, \| f_n \|_{L^2} .$$

Thus, that if we pass to the limit we obtain the desired result. ■

In the following theorem we prove a formula of inversion, due to Plancherel.

Theorem 4.4.4 *If the functions $f, g \in L^2(R)$ then we have the following formula of inversion*

$$\int_{-\infty}^{+\infty} \widehat{f}(x)\overline{\widehat{g}(x)}\mathrm{d}x = 2\pi \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\mathrm{d}x.$$

Proof According to the Parseval's formula, we can write

$$\left\| \widehat{f} + \widehat{g} \right\|_{L^2}^2 = 2\pi \, \| f + g \|_{L^2}^2 ,$$

that is,

$$\int_{-\infty}^{+\infty} \left(\widehat{f}(x) + \widehat{g}(x)\right) \left(\overline{\widehat{f}(x)} + \overline{\widehat{g}(x)}\right) \mathrm{d}x =$$
$$= 2\pi \int_{-\infty}^{+\infty} \left(f(x) + g(x)\right) \left(\overline{f(x)} + \overline{g(x)}\right) \mathrm{d}x.$$

After simple calculations, again, be using the Parseval's formula, it follows

$$\int_{-\infty}^{+\infty} \widehat{f}(x)\overline{\widehat{g}(x)}\mathrm{d}x + \int_{-\infty}^{+\infty} \widehat{g}(x)\overline{\widehat{f}(x)}\mathrm{d}x =$$
$$= 2\pi \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\mathrm{d}x + 2\pi \int_{-\infty}^{+\infty} \overline{f(x)}g(x)\mathrm{d}x. \tag{4.4.9}$$

We make these calculations substituting g by ig and then

$$-\int_{-\infty}^{+\infty}\widehat{f}(x)\overline{\widehat{g}(x)}\mathrm{d}x+i\int_{-\infty}^{+\infty}\overline{\widehat{f}(x)}\widehat{g}(x)\mathrm{d}x=$$
$$=-2\pi i\int_{-\infty}^{+\infty}f(x)\overline{g(x)}\mathrm{d}x+2\pi i\int_{-\infty}^{+\infty}\overline{f(x)}g(x)\mathrm{d}x.$$

Here we simplify by $(-i)$ and add the resulting equality to the equality (4.4.9). So, we obtain the result of Plancherel. ■

Another formula of inversion is the result from the following theorem.

Theorem 4.4.5 *If the functions $f, g \in L^2(R)$ then we have the following formula of inversion*

$$\int_{-\infty}^{+\infty}\widehat{f}(x)g(x)\mathrm{d}x=\int_{-\infty}^{+\infty}f(x)\widehat{g}(x)\mathrm{d}x. \tag{4.4.10}$$

Proof Since the functions $f, g \in L^2(R)$ we will deduce that we can attach the truncated functions, respectively f_n and g_k. We already proved that $f_n, g_k \in L^1(R)$ and then we can attach its transforms, in the sense of the functions from $L^1(R)$. Using the Fourier's transform for the truncated functions, it results

$$\int_{-\infty}^{+\infty}\widehat{f_n}(x)g_k(x)\mathrm{d}x=\int_{-\infty}^{+\infty}g_k(x)\left\{\int_{-\infty}^{+\infty}f_n(t)e^{ixt}\mathrm{d}t\right\}\mathrm{d}x=$$
$$=\int_{-\infty}^{+\infty}f_n(x)\left\{\int_{-\infty}^{+\infty}g_k(t)e^{ixt}\mathrm{d}x\right\}\mathrm{d}t=\int_{-\infty}^{+\infty}f_n(t)\widehat{f_k}(t)\mathrm{d}t. \tag{4.4.11}$$

In these calculations we have inverted the order of integration because the integrals are computed on finite intervals, taking into account the definition of the truncated functions.

The equality (4.4.11) proves that the formula of inversion (4.4.10) is still valid for the truncated functions. We now fix f_n and use the result from the Theorem 4.4.2, such that the sequence $\{\widehat{g_k}\}$ is convergent, piecewise, in the sense of L^2, to a function from L^2. Similarly it follows the fact that the sequence $\{\widehat{f_n}\}$ is convergent, piecewise, in the sense of L^2, to a function from L^2. We use then the fact that the two limits are g and respectively f. Now, based on the Lebesque's theorem with regard to pass to the limit under the integral, from (4.4.11) it results (4.4.10), that is, the theorem is proved. ■

At the end of this paragraph, we prove the last formula of inversion for the Fourier's transform.

Theorem 4.4.6 *We consider the function $f \in L^2(R)$ and define the function g by*

$$g(x)=\overline{\widehat{f}(x)},\ \forall x\in R.$$

Then

$$f(x) = \frac{1}{2\pi}\overline{\widehat{g}(x)},\ \forall x \in R.$$

Proof According to the definition of the norm in L^2, we have

$$\begin{aligned}\left\|f - \frac{1}{2\pi}\overline{\widehat{g}}\right\|_{L^2}^2 = &\int_{-\infty}^{+\infty}\left(f(x) - \frac{1}{2\pi}\overline{\widehat{g}(x)}\right)\left(\overline{f(x)} - \frac{1}{2\pi}\widehat{g}(x)\right)\mathrm{d}x = \\ &= \|f\|_{L^2}^2 + \frac{1}{4\pi^2}\|\widehat{g}\|_{L^2}^2 - \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\mathrm{d}x - \\ &- \frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{f(x)\widehat{g}(x)}\mathrm{d}x.\end{aligned} \tag{4.4.12}$$

By using two times the Parseval's formula, we obtain

$$\begin{aligned}&\frac{1}{4\pi^2}\|\widehat{g}\|_{L^2}^2 = \frac{2\pi}{4\pi^2}\|g\|_{L^2}^2 = \\ &= \frac{1}{2\pi}\left\|\overline{\widehat{f}}\right\|_{L^2}^2 = \frac{1}{2\pi}\left\|\widehat{f}\right\|_{L^2}^2 = \frac{1}{2\pi}\|f\|_{L^2}^2.\end{aligned} \tag{4.4.13}$$

On the other hand, using Eq. (4.4.10) and then the Parseval's formula, it results

$$\begin{aligned}-\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\mathrm{d}x &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{f}(x)g(x)\mathrm{d}x = \\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{f}(x)\overline{\widehat{f}(x)}\mathrm{d}x = -\frac{1}{2\pi}\left\|\widehat{f}\right\|_{L^2}^2 = \\ &= -\frac{2\pi}{2\pi}\|f\|_{L^2}^2 = -\|f\|_{L^2}^2.\end{aligned} \tag{4.4.14}$$

Similarly,

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{\widehat{g}(x)}\widehat{f}(x)\mathrm{d}x = \overline{-\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\mathrm{d}x} = \tag{4.4.15}$$

$$= \overline{-\|f\|_{L^2}^2} = -\|f\|_{L^2}^2.$$

Here we used the result from Eq. (4.4.14) and the fact that the conjugated of a real number is equal to itself.

If we take into account formulas (4.4.13)–(4.4.15) in (4.4.12), we obtain the formula from our theorem. ∎

Chapter 5
Calculus of Variations

5.1 Introduction

The modern engineer often has to deal with problems that require a sound mathematical background and set skills in the use of various mathematical methods. Expanding the mathematical outlook of engineers contributes appreciably to new advances in technology. The calculus of variations is one of the most important divisions of classical mathematical analysis in regards to applications.

At the beginning of this paragraph we remember some basic elementary notions of the classical mathematical analysis that we will use in this chapter.

1. A linear space R is a *normed linear space* if to every element $x \in R$ there is associated a nonnegative real number $\|x\|$ called the *norm* of that element, and:

(i) $\|x\| = 0$ only when $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangle axiom for norms).

2. A set M of elements $x, y, z, \ldots$ of any nature whatsoever is a *metric space* if to each pair of elements $x, y \in M$ there is associated a nonnegative real number $\varrho(x, y)$ such that

(i) $\varrho(x, y) = 0$ if and only if $x = y$ (identity axiom);
(ii) $\varrho(x, y) = \varrho(y, x)$ (symmetry axiom);
(iii) $\varrho(x, y) + \varrho(y, z) \geq \varrho(x, z)$ (triangle axiom).

The number $\varrho(x, y)$ is called the *distance between the elements x and y*. Of course, every normed linear space is a metric space. This statement can be immediately argued if we put $\varrho(x, y) = \|x - y\|$.

3. The *space* $C^0[a, b]$ is the space of all functions $y(x)$ continuous on $[a, b]$. The usual norm for each element $y \in C^0[a, b]$ is

$$\|y\|_0 = \max_{a \leq x \leq b} |y(x)|,$$

M. Marin and A. Öchsner, *Complements of Higher Mathematics*,
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where $|y(x)|$ is the modulus of the element $y(x) \in C^0[a, b]$.

4. The *space* $C^1[a, b]$ is the space of all functions $y(x)$ continuous on $[a, b]$ together with their first derivatives. The usual norm for each element $y \in C^1[a, b]$ is

$$\|y\|_1 = \max_{a \leq x \leq b} |y(x)| + \max_{a \leq x \leq b} |y'(x)|,$$

where $y'(x)$ is the first derivative of function $y(x)$.

5. The *space* $C^n[a, b]$ is the space of all functions $y(x)$ continuous on $[a, b]$ together with their derivatives up to order n inclusive (n is a fixed natural number). The usual norm for each element $y \in C^n[a, b]$ is

$$\|y\|_n = \sum_{k=0}^{n} \max_{a \leq x \leq b} |y^{(k)}(x)|,$$

where $y^{(k)}(x)$ is the derivative of k-th order of function $y(x)$. Sometimes the norm of the element $y(x) \in C^n[a, b]$ is defined as follows

$$\|y\| = \max_{a \leq x \leq b} \left\{ |y(x)|, |y'(x)|, \ldots, |y^{(n)}(x)| \right\}.$$

Suppose we have a certain class M of functions $y(x)$. If to each function $y(x) \in M$ there is associated, by some law, a definite number J, then we say that a functional J is defined in the class M and we write $J = J(y(x))$.

The class M of functions $y(x)$ on which the functional $J = J(y(x))$ is defined is called the *domain of definition of the functional.*

Example 5.1. Let $M = C^1[a, b]$ be the class of functions $y(x)$ that have continuous derivatives on the interval $[a, b]$ and let

$$J(y(x)) = y'(x_0), \ \ x_0 \in [a, b].$$

It is clear that $J = J(y(x))$ is a functional defined in the indicated class of functions because to each function of this class there is associated a definite number, namely, the value of the derivative of the function at the fixed point x_0.

If, for instance, $a = 1, \ b = 3$ and $x_0 = 2$, then for $y(x) = x^2 + 1$ we get

$$J(x^2) = 2x|_{x=2} = 4.$$

For $y(x) = \ln(1 + x)$ we have

$$J(\ln(1 + x)) = \left. \frac{1}{1 + x} \right|_{x=2} = \frac{1}{3}.$$

Example 5.2. Let $M = C^0[-1, 1]$ be the class of functions $y(x)$ continuous on the interval $[-1, 1]$ and let $\varphi(x, y)$ be a given function defined and continuous for all $-1 \leq x \leq 1$ and for all real y. Then

$$J(y(x)) = \int_{-1}^{1} \varphi(x, y(x))\mathrm{d}x,$$

will be a functional defined on the indicated class of functions. For instance, if $\varphi(x, y) = x/(1 + y^2)$, then for $y(x) = x$ it follows

$$J(x) = \int_{-1}^{1} \frac{x}{1 + x^2}\mathrm{d}x = 0,$$

and for $y(x) = 1 + x$ we have

$$J(1 + x) = \int_{-1}^{1} \frac{x}{1 + (1 + x)^2}\mathrm{d}x = \ln\sqrt{5} - \arctan 2.$$

Example 5.3. Let $M = C^1[a, b]$ be the class of functions $y(x)$ having continuous derivatives $y'(x)$ on the interval $[a, b]$. Then

$$J(y(x)) = \int_{a}^{b} \sqrt{1 + y'^2(x)}\mathrm{d}x,$$

is a functional defined on that class of functions. This functional geometrically describes the arc length of the curve $y = y(x)$ with ends at the points $A(a, y(a))$ and $B(b, y(b))$.

Definition 5.1.1 We say that the curves $y = y(x)$ and $y = y_1(x)$ specified on the interval $[a, b]$ are close in the sense of vicinity of the zero-th order if $|y(x) - y_1(x)|$ is small on $[a, b]$.

Geometrically, this means that those curves on the interval $[a, b]$ are *close in regards to the coordinates*.

We say that the curves $y = y(x)$ and $y = y_1(x)$ specified on the interval $[a, b]$ are close in the sense of vicinity of the first order if $|y(x) - y_1(x)|$ and $\left|y'(x) - y_1'(x)\right|$ are small on $[a, b]$. Geometrically, this means that those curves on the interval $[a, b]$ are close both in regards to the ordinates and to the directions of the tangents at the appropiate points. More generally, we say that he curves $y = y(x)$ and $y = y_1(x)$ specified on the interval $[a, b]$ are close in the sens of vicinity of the k-th order if the moduli

$$|y(x) - y_1(x)|, \ \left|y'(x) - y_1'(x)\right|, \ldots \left|y^{(k)}(x) - y_1^{(k)}(x)\right|$$

are small on $[a, b]$.

Of course, if two curves are close in the sense of vicinity of the k-th order, they are closer in the sense of vicinity of any smaller order.

Example 5.1. For n sufficiently large, we define on the interval $[0, \pi]$ the following functions

$$y(x) = \frac{\sin n^2 x}{n}, \ y_1(x) = 0, \ \forall x \in [0, \pi].$$

These functions are close in the sense of vicinity of zero-th order since

$$|y(x) - y_1(x)| = \left|\frac{\sin n^2 x}{n}\right| \le \frac{1}{n}.$$

That is to say, on the entire interval $[0, \pi]$ this difference is small in modulus if n is sufficiently large. But, there is no vicinity of the first order since

$$\left|y'(x) - y_1'(x)\right| = n \left|\cos n^2 x\right|$$

and, for instance, at the points

$$x = \frac{2\pi}{n^2},$$

we have $\left|y'(x) - y_1'(x)\right| = n$ and, hence, $\left|y'(x) - y_1'(x)\right|$ can be made arbitrarily large for sufficiently large n.

Example 5.2. For n sufficiently large, we define on the interval $[0, \pi]$ the following functions

$$y(x) = \frac{\sin n^2 x}{n^2}, \ y_1(x) = 0, \ \forall x \in [0, \pi].$$

These functions are close in the sense of vicinity of the first order since

$$|y(x) - y_1(x)| = \left|\frac{\sin n^2 x}{n^2}\right| \le \frac{1}{n^2}$$

and

$$\left|y'(x) - y_1'(x)\right| = \left|\frac{\cos nx}{n}\right| \le \frac{1}{n},$$

are small, for sufficiently large n.

Definition 5.1.2 The distance between the curves $y = f(x)$ and $y = f_1(x)$ $(a \le x \le b)$, where $f(x)$ and $f_1(x)$ are functions continuous on $[a, b]$, is a nonnegative number ϱ equal to the maximum of $|f_1(x) - f(x)|$ on the interval $a \le x \le b$:

$$\varrho = \varrho\,[f_1(x), f(x)] = \max_{a \le x \le b} |f_1(x) - f(x)|$$

The n-th order distance between the curves $y = f(x)$ and $y = f_1(x)$ $(a \le x \le b)$, where the functions $f(x)$ and $f_1(x)$ have continuous derivatives of order n on the interval $[a, b]$, is the largest of the maxima of the quantities

$$|f_1(x) - f(x)|\,,\ \left|f_1'(x) - f'(x)\right|, \ldots, \left|f_1^{(n)}(x) - f^{(n)}(x)\right|$$

on the interval $[a, b]$. We will denote this distance as follows

$$\varrho_n = \varrho_n\,[f_1(x), f(x)] = \max_{0 \le k \le n} \max_{a \le x \le b} \left|f_1^{(k)}(x) - f^{(k)}(x)\right|.$$

Example 5.1. Find the first order distance between the curves $f(x) = x^2$ and $f_1(x) = x^3$ on the interval [0, 1].

We find the derivatives of the given functions, $f'(x) = 2x$ and $f_1'(x) = 3x^2$, and we consider the functions

$$y_1(x) = x^2 - x^3,\ \ y_2(x) = 2x - 3x^2.$$

Let us find their maximum values on the interval [0, 1]. We have $y_1'(x) = 2x - 3x^2$. Equating this derivative to zero, we find the stationary points of the function $y_1(x)$: $x_1 = 0,\ x_2 = 2/3$. Furthermore,

$$y_1|_{x=0} = 0,\ \ y_1|_{x=\frac{2}{3}} = \frac{4}{27}.$$

The value of $y_1(x)$ at the endpoint is $y_1(1) = 0$, whence

$$\varrho_0 = \max_{0 \le x \le 1} \left|x^3 - x^2\right| = \max_{0 \le x \le 1} \left(x^2 - x^3\right) = \frac{4}{27}.$$

Let us now find the zero-order distance $\tilde{\varrho}$ between the derivatives $f'(x) = 2x$ and $f_1'(x) = 3x^2$:

$$\tilde{\varrho} = \max_{0 \le x \le 1} \left|y_2'(x)\right| = \max_{0 \le x \le 1} \left|2x - 3x^2\right|.$$

If we construct the graph of the function $\left|2x - 3x^2\right|$ then it is evident that $\tilde{\varrho} = 1$. Thus, the first order distance ϱ_1 between the curves $f(x) = x^2$ and $f_1(x) = x^3$ is equals to

$$\varrho_1 = \max\,(\varrho_0, \tilde{\varrho}) = 1.$$

Definition 5.1.3 The n-th order ε-neighbourhood of a curve $y = f(x)$ $(a \le x \le b)$ is defined as the collection of curves $y = f_1(x)$ whose n-th order distances from the curve $y = f(x)$ are less than ε:

$$\varrho_n = \varrho_n \left[f_1(x), f(x) \right] < \varepsilon.$$

A zero-th ε-neighborhood is called a *strong ε-neighborhood of the function $y = f(x)$*.

The strong ε-neighborhood of the curve $y = f(x)$ consists of curves located in a strip of width 2ε about the curve $y = f(x)$.

A first order ε-neighborhood is called a *weak ε-neighborhood of the function $y = f(x)$*.

Definition 5.1.4 A functional $J(y(x))$ defined in a class M of functions $y(x)$ is said to be continuous for $y = y_0(x)$ in the sense of n-th order vicinity if for any number ε there exists a number $\eta > 0$ such that for admissible functions $y = y(x)$ satisfying the conditions

$$|y(x) - y_0(x)| < \eta, \ \left| y'(x) - y_0'(x) \right| < \eta, \ldots, \left| y^{(n)}(x) - y_0^{(n)}(x) \right| < \eta,$$

the inequality $|J\left(y(x)\right) - J\left(y_0(x)\right)| < \varepsilon$.

In other words,

$$\varrho_n \left[y(x), y_0(x) \right] < \eta \Rightarrow |J\left(y(x)\right) - J\left(y_0(x)\right)| < \varepsilon.$$

A functional that is not continuous in the sense of $n - th$ order vicinity will be called *discontinuous* in the sense of the indicated vicinity. Putting

$$y^{(k)}(x) = y_0^{(k)}(x) + \alpha\omega^{(k)}(x), \ k = 0, 1, 2, \ldots, n$$

where α is some parameter and $\omega(x)$ is an arbitrary function in the class M, we note that

$$\lim_{\alpha \to 0} y^{(k)}(x) = y_0^{(k)}(x), \ k = 0, 1, 2, \ldots, n$$

and the definition of the continuity of the functional when $y(x) = y_0(x)$ may be written as:

$$\lim_{\alpha \to 0} J\left[y_0(x) + \alpha\omega(x) \right] = J\left[y_0(x) \right].$$

Example 5.1. Let us show that the functional

$$J(y(x)) = \int_0^1 \left[y(x) + 2y'(x) \right] \mathrm{d}x$$

defined in the space $C^1[0, 1]$ is continuous on the function $y_0(x) = x$ in the sense of first order vicinity.

Indeed, take an arbitrary number $\varepsilon > 0$ and show that there exists a number $\eta > 0$ such that $|J(y(x)) - J(x)| < \varepsilon$ as soon as $|y(x) - x| < \eta$ and $|y'(x) - 1| < \eta$. We have

$$|J(y(x)) - J(x)| = \left| \int_0^1 \left[y(x) + 2y'(x) - x - 2\right] \mathrm{d}x \right| \leq$$

$$\leq \int_0^1 |y(x) - x| \mathrm{d}x + 2 \int_0^1 \left|y'(x) - 1\right| \mathrm{d}x.$$

We choose $\eta = \varepsilon/3$. Then for all $y(x) \in C^1[0, 1]$ for which

$$|y(x) - x| < \frac{\varepsilon}{3} \textit{ and } \left|y'(x) - x\right| < \frac{\varepsilon}{3}$$

we will have

$$|J(y(x)) - J(x)| < \varepsilon.$$

Thus, for every $\varepsilon > 0$ there exists an $\eta > 0$, for example, $\eta = \varepsilon/3$, such that as soon as $\varrho_1[y(x), x] < \eta$ then $|J(y(x)) - J(x)| < \varepsilon$.

By definition, this means that the given functional is continuous on the function $y_0(x) = x$ in the sense of first order vicinity. In fact, it is easy to see that this functional is continuous in the sense of first order vicinity on any curve $y(x) \in C^1[0, 1]$.

Example 5.2. Let us show that the functional

$$J(y(x)) = \int_0^1 \left[y(x) + 2y'(x)\right] \mathrm{d}x$$

defined in the space $C^1[0, \pi]$ is discontinuous on the function $y_0(x) = 0$ in the sense of zero-th order vicinity.

Indeed, let $y_0(x) = 0$ on $[0, \pi]$ and $y_n(x) = (\sin nx)/n$. Then $\varrho_0[y_0(x), y_n(x)] = 1/n$ and $to 0$ as $n \to \infty$. On the other hand, the difference

$$J\left(y_n(x)\right) - J\left(y_0(x)\right) = \int_0^\pi \frac{\cos^n x}{n} \mathrm{d}x = \frac{\pi}{2}$$

does not depend on n. Thus, as $n \to \infty$, $J\left(y_n(x)\right)$ does not tend to $J\left(y_0(x)\right)$, and, hence, the given functional is discontinuous in the sense of zero-th order vicinity on the function $y_0(x)$.

But it is easy to prove that the functional under consideration is continuous on the function $y_0(x) = 0$ in the sense of first order vicinity.

Definition 5.1.5 Let M be a normed linear space of the functions $y(x)$. The functional $L(y(x))$ defined in the space M is called a linear functional if it satisfies the following two conditions

(1) $L(cy(x)) = cL(y(x))$,
where c is an arbitrary constant;
(2) $L(y_1(x) + y_2(x)) = L(y_1(x)) + L(y_2(x))$,
where $y_1(x) \in M$ and $y_2(x) \in M$.

Definition 5.1.6 If the increment in the functional

$$\Delta J = J(y(x) + \delta y) - J(y(x))$$

can be represented as

$$\Delta J = L(y(x), \delta y) + \beta(y(x), \delta y)\|\delta y\|,$$

where $L(y(x), \delta y)$ is a linear functional with respect to δy and $\beta(y(x), \delta y) \to 0$ as $\|\delta y\| \to 0$, then the portion of the increment of the functional that is linear with respect to δy, that is, $L(y(x), \delta y)$ is called the variation of the functional and is denoted by δJ. In this case the functional $J(y(x))$ is said to be differentiable at the point $y(x)$.

Example 5.1. Let us show that the functional

$$J(y(x)) = \int_a^b y(x)\mathrm{d}x$$

specified in the space $C^0[a, b]$ is differentiable at each point $y(x)$ of that space.
Indeed, by direct calculations

$$\Delta J = J(y + \delta y) - J(y) =$$

$$= \int_a^b [y(x) + \delta y(x)]\mathrm{d}x - \int_a^b y(x)\mathrm{d}x = \int_a^b \delta y(x)\mathrm{d}x.$$

Thus,

$$\Delta J = \int_a^b \delta y(x)\mathrm{d}x.$$

This is a linear functional with respect to $\delta y(x)$. In the given case the entire increment of the functional reduced to the linear functional with respect to $\delta y(x)$. This functional is differentiable at every point $y(x)$ and its variation is

$$\delta J = \int_a^b \delta y(x)\mathrm{d}x.$$

Now, we give a second definition of the variation of a functional.

Definition 5.1.7 The variation of the functional $J(y(x))$ at the point $y = y(x)$ is defined as the value of the derivative of the functional $J(y(x)+\alpha\delta y(x))$ with respect to the parameter α when $\alpha = 0$:

$$\delta J = \frac{\partial}{\partial\alpha} J(y(x) + \alpha\delta y(x))\bigg|_{\alpha=0}.$$

If the variation of a functional exists as the principal linear part of its increment, that is, in the sense of the first definition, then the variation also exists as the value of the derivative with respect to the parameter α when $\alpha = 0$, and these variations coincide.

Example 5.1. Using the second definition, find the variation of the functional

$$J(y(x)) = \int_a^b y^2(x)\mathrm{d}x.$$

Firstly, the variation of this functional in the sense of the first definition is equal to

$$\delta u = 2\int_a^b y(x)\delta y(x)\mathrm{d}x.$$

Let us find the variation of our functional using the second definition of a variation. We have

$$J(y(x) + \alpha\delta y(x)) = \int_a^b [y(x) + \alpha\delta y(x)]^2\mathrm{d}x.$$

Then

$$\frac{\partial}{\partial\alpha} J(y(x) + \alpha\delta y(x)) = 2\int_a^b [y(x) + \alpha\delta y(x)]\delta y(x)\mathrm{d}x,$$

and, consequently,

$$\delta J = \frac{\partial}{\partial\alpha} J(y(x) + \alpha\delta y(x))\bigg|_{\alpha=0} = 2\int_a^b y(x)\delta y(x)\mathrm{d}x.$$

So, it is easy to see that the variations of the functional under considerations in the sense of first and second definitions coincide.

Remark. The second definition of a variation of a functional is somewhat broader than the first in the sense that there are functionals from the increments of which it is not possible to isolate a principal linear part, but the variation exists in the sense of the second definition. We will demonstrate this fact using the example of functions for which the formulated assertion is equivalent to the fact that the existence of derivatives in any direction is not sufficient for the existence of a differential of the function.

So, let us consider the function

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} = \frac{\varrho}{2} \sin 2\varphi, \;\; x^2 + y^2 \neq 0,$$

where ϱ and φ are the polar coordinates of the point (x, y).

The partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at every point and at the origin are equal to zero, but the differential $\mathrm{d}f$ does not exists at the origin. Indeed, given the existence of $\mathrm{d}f$, the gradient of the function f at the origin would in this case be equal to zero, and therefore a derivative in any direction, $\mathrm{d}f(0, 0)/\mathrm{d}l$, would be equal to zero. Yet, as can readily be seen

$$\frac{\mathrm{d}f(0, 0)}{\mathrm{d}l} = \frac{1}{2} \sin 2\varphi,$$

which is, generally speaking, different from zero. Here, φ is the angle formed by the vector l with the x-axis and the derivative in the direction of l coincides with the derivative with respect to ϱ.

A functional $J(x, y)$ dependent on two elements x and y (lying in a certain linear space) is called a *bilinear functional* if, for a fixed x, it is a linear functional of y and, for a fixed y, it is a linear functional of x. Thus, the functional $J(x, y)$ is bilinear if

$$J(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 J(x_1, y) + \alpha_2 J(x_2, y),$$

$$J(x, \alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 J(x, y_1) + \alpha_2 J(x, y_2).$$

Putting $y = x$ in a bilinear functional, we get the expresion $J(x, x)$, which is called a *quadratic functional*. A bilinear functional in a finite-dimensional space is called a *bilinear form*. A quadratic functional $J(x, x)$ is said to be *positive definite* if $J(x, x) > 0$ for any nonzero element x.

Example 5.1. If $A(t)$ is a fixed continuous function, then the expression

$$J(x, y) = \int\limits_a^b A(t)x(t)y(t)\mathrm{d}t$$

is a bilinear functional, and the expression

$$\int_a^b A(t)x^2(t)\mathrm{d}t$$

is a quadratic functional in the space $C^0[a, b]$.

If $A(t) > 0$ for all $t \in [a, b]$, then this quadratic functional will be positive definite.

Example 5.2. If $A(t)$, $B(t)$ and $C(t)$ are fixed continuous functions, then the expression

$$\int_a^b \left[A(t)x^2(t) + B(t)x(t)x'(t) + C(t)x'^2(t)\right] \mathrm{d}t$$

is an example of a quadratic functional defined for all functions in the space $C^1[a, b]$.

Now, we intend to introduce the second variation of a functional.

Definition 5.1.8 Let $J(y)$ be a functional defined in some normed linear space. We will say that the functional $J(y)$ has a second variation if its increment

$$\Delta J = J(y + \delta y) - J(y),$$

may be written in the form

$$\Delta J = L_1(\delta y) + \frac{1}{2}L_2(\delta y) + \beta\|\delta y\|^2,$$

where $L_1(\delta y)$ is a linear functional, $L_2(\delta y)$ is a quadratic functional, and $\beta \to 0$ as $\|\delta\| \to 0$.

We will call the quadratic functional $L_2(\delta y)$ the second variation, or second differential, of the functional $J(y)$ and denote it by $\delta^2 J$.

We can prove that if the second variation of a functional exists, then it is uniquely defined.

Example 5.1. Let us compute the second variation of the following functional

$$J(y) = \int_0^1 \left(xy^2 + y'^2\right) \mathrm{d}x.$$

Using the definition, we obtain

$$\Delta J = J(y + \delta y) - J(y) =$$

$$= \int_0^1 \left[x(y+\delta y)^2 + (y'+\delta y')^3 - xy^3 - y'^3\right) \mathrm{d}x =$$

$$= \int_0^1 \left[2xy\delta y + x(\delta y)^2 + 3y'^2\delta y' + 3y'(\delta y')^2 + (\delta y')^3\right] \mathrm{d}x =$$

$$= \int_0^1 \left(2xy\delta y + 3y'^2\delta y'\right) \mathrm{d}x + \int_0^1 \left[x(\delta y)^2 + 3y'(\delta y')^2\right) \mathrm{d}x + \int_0^1 \left(\delta y'\right)^3 \mathrm{d}x.$$

For a fixed $y(x)$, the first term in the right-hand member of the above relation is a functional linear with respect to $\delta y(x)$; the second term of the right-hand member is a quadratic functional. Finally, the third term of the right-hand member allows for the obvious estimate

$$\left|\int_0^1 \left(\delta y'\right)^3 \mathrm{d}x\right| \leq \left(\max\left|\delta y'\right|\right)^2 \int_0^1 \left|\delta y'\right| \mathrm{d}x \leq \|\delta y\|^2 \int_0^1 \left|\delta y'\right| \mathrm{d}x,$$

(the norm in the sense of the space $C^1[0, 1]$), whence it is seen that this term can be represented in the form $\beta \|\delta y\|^2$, where $\beta \to 0$ as $\|\delta y\| \to 0$. According to the definition, the given functional has the second variation $\delta^2 J$ and it is equal to

$$\delta^2 J = 2\int_0^1 \left[x(\delta y)^2 + 3y'(\delta y')^2\right) \mathrm{d}x.$$

Let us state and demonstrate the necessary condition for the extremum of a functional.

Definition 5.1.9 We say that a functional $J(y(x))$ attains a maximum on a curve $y = y_0(x)$ if the values of the functional $J(y(x))$ on any curve close to $y = y_0(x)$ do not exceed $J(y_0(x))$, that is,

$$\Delta J = J(y(x)) - J(y_0(x)) \leq 0.$$

If $\Delta J \leq 0$ and $\Delta J = 0$ only when $y(x) = y_0(x)$, then we say that a strict maximum is attained on the curve $y = y_0(x)$. The curve $y = y_0(x)$ on which a minimum is attained can be defined in a similar way. In this case $\Delta J \geq 0$ on all curves close to the curve $y = y_0(x)$.

Example 5.1. Let us show that the functional

$$J(y(x)) = \int_0^1 \left(x^2 + y^2\right) \mathrm{d}x$$

attains a strict minimum on the curve $y(x) = 0$.

Indeed, for any functions $y(x)$ continuous on [0, 1] we have

$$\Delta J = J(y(x)) - J(0) = \int_0^1 \left(x^2 + y^2\right) \mathrm{d}x - \int_0^1 x^2 \mathrm{d}x = \int_0^1 y^2 \mathrm{d}x \geq 0,$$

equality occuring only when $y(x) = 0$.

5.2 Euler's Equation

Using a simplified language, the object of study for calculus of variations is to find the minimum or maximum value of a functional. The results in this division of mathematics are more general as in the classical theory of functions. But, many results are equivalent with the corresponding result in the case of functions.

The problems of variational calculus are originated from physics, but in the last decade the sphere of the problems contains other divisions of sciences.

It is considered that first problem of variational calculus is due to Jean Bernoulli who has stated in 1696 the following problem.

Consider two points A and B which are not lying on the same vertical and, also, not on the same horizontal. A power material point falls down on a curve with its end in A and B. Determine the trajectory of the material point which starts from the point A such that the time necessary to attain the point B is minimum.

This problem is called the *brahistrocronic curve* and has been solved after many years, by Leibniz, Newton and Jacob Bernoulli.

In mechanics, the Hamilton's principle asserts that a mechanical system chooses, from all possible trajectories in a certain period of time, that trajectory on which the mechanical work is minimum.

Let us specify other mathematical problems which need the techniques of variational calculus to find their solutions.

1. Geodesics of surfaces. Given is a surface in the Euclidean space and two arbitrary points A and B lying in the surface. Find the arc of the curve contained in the surface that connects A and B in such a way that its length is smallest.

2. Surface of minimum area. Given are two arbitrary points A and B. Find the arc of the curve that connects A and B in such a way that the area obtained by rotating this arc about the x-axis is smallest.

3. Isoperimetric problem. Let A and B be two arbitrary points. In the collection of the curves having the same length l that connect the points A and B, find one that together with the segment AB closes a surface of maximum area.

In the following we specify the necessary notions to solve a problem of variational calculus. We intend to find the extremum only for a functional of the integral type. The set of definition for the functional will be the space $C^1[a, b]$, already defined as

$$C^1[a, b] = \left\{y = y(x),\ y : [a, b] \to R : y \text{ is of class } C^1\right\}.$$

This space is endowed by a structure of linear space with regard to the usual operations of functions and with a structure of normed space with regard to the norm

$$\|y\|_1 = \max_{a \le x \le b} |y(x)| + \max_{a \le x \le b} |y'(x)|.$$

The integrant of the functional will be a function L that depends of three variables and will be called the *Lagrangean of the functional*:

$$L : [a, b] \times R \times R \to R,\ L = L(x, y(x), y'(x)).$$

Now, we can define the functional, denoted by I, for that we will find the extreme value (minimum or maximum):

$$I : C^1[a, b] \to R,\ I(y) = \int_a^b L(x, y(x), y'(x))\mathrm{d}x. \tag{5.2.1}$$

So, in short, a problem of variational calculus consists in finding of that function of C^1 class that gives the extreme value (minimum or maximum) of the functional I.

Let us fixe the values of the function y at the ends of the interval, i.e. $y(a) = y_a$ and $y(b) = y_b$, where y_a and y_a are real known numbers.

In the following lemma we prove an auxiliary, but very important, result, called the *fundamental lemma of variational calculus*.

Lemma 5.2.1 *Consider $y = y(x)$ a function such that $y \in C^1[a, b]$. If*

$$\int_a^b y(x)\eta(x)\mathrm{d}x = 0$$

for any function $\eta = \eta(x)$ such that $\eta \in C^1[a, b]$ and $\eta(a) = \eta(b) = 0$, then $y(x) = 0,\ \forall x \in [a, b]$.

Proof We suppose, on the contrary, that $\exists x_0 \in [a, b]$ such that $y(x_0) \neq 0$. Without loss of generality, we can suppose that $y(x_0) > 0$. The proof is similarly if we should that $y(x_0) < 0$. Since the function $y(x)$ is continuous, according to a classical

theorem, the values of the function $y(x)$ are strict positive in a whole vicinity of x_0, that is

$$y(x) > 0,\ \forall x \in [x_0 - \varepsilon, x_0 + \varepsilon]\,,$$

where ε is a positive real number, arbitrarily small.

Because the integral of the hypothesis is null for any function η, we deduce that statement is true also for a particular η. So, we define the function

$$\eta_0(x) = \begin{cases} 0,\ x \in [a, x_0 - \varepsilon), \\ \left[(x - x_0)^2 - \varepsilon^2\right]^2,\ x \in [x_0 - \varepsilon, x_0 + \varepsilon], \\ 0,\ x \in (x_0 + \varepsilon, b]. \end{cases}$$

Clearly, we have (by definition) $\eta_0(a) = \eta_0(b) = 0$. Let us prove that $\eta_0(x) \in C^1[a, b]$. It is easy to see that

$$\lim_{x \nearrow x_0 - \varepsilon} \eta_0(x) = \lim_{x \searrow x_0 - \varepsilon} \eta_0(x) = \eta_0(x_0 - \varepsilon),$$

such that η_0 is a continuous function at the point $x_0 - \varepsilon$. Similarly,

$$\lim_{x \nearrow x_0 + \varepsilon} \eta_0(x) = \lim_{x \searrow x_0 + \varepsilon} \eta_0(x) = \eta_0(x_0 + \varepsilon),$$

such that η_0 is a continuous function at the point $x_0 + \varepsilon$. By direct calculations, we have

$$\eta_0'(x) = \begin{cases} 0,\ x \in [a, x_0 - \varepsilon), \\ 4(x - x_0)\left[(x - x_0)^2 - \varepsilon^2\right],\ x \in [x_0 - \varepsilon, x_0 + \varepsilon], \\ 0,\ x \in (x_0 + \varepsilon, b]. \end{cases}$$

Then, we have

$$\lim_{x \nearrow x_0 - \varepsilon} \eta_0'(x) = \lim_{x \searrow x_0 - \varepsilon} \eta_0'(x) = 0$$

and, also

$$\lim_{x \nearrow x_0 + \varepsilon} \eta_0'(x) = \lim_{x \searrow x_0 + \varepsilon} \eta_0'(x) = 0$$

such that we conclude that $\eta_0(x) \in C^1[a, b]$. Therefore, we must have

$$\int_a^b y(x)\eta_0(x)\mathrm{d}x = 0 \Rightarrow$$

$$\Rightarrow \int_a^{x_0-\varepsilon} y(x)\eta_0(x)\mathrm{d}x + \int_{x_0-\varepsilon}^{x_0+\varepsilon} y(x)\eta_0(x)\mathrm{d}x + \int_{x_0+\varepsilon}^{b} y(x)\eta_0(x)\mathrm{d}x = 0 \Rightarrow$$

$$\Rightarrow \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} y(x)\left[(x-x_0)^2-\varepsilon^2\right]^2 \mathrm{d}x = 0.$$

But, this is a contradiction, because the integrant of this integral is strict positive,

$$y(x) > 0,\ \forall x \in [x_0-\varepsilon, x_0+\varepsilon],\ \left[(x-x_0)^2-\varepsilon^2\right]^2 > 0,\ \forall x \in [x_0-\varepsilon, x_0+\varepsilon].$$

So, the lemma is concluded.

In the following theorem we prove a necessary condition for the extremum of a functional. This fundamental result is due to Cauchy and it is very important for the whole variational calculus.

Theorem 5.2.1 *If $y(x)$ is the function where the functional $I(y)$ (defined in Eq. (5.2.1)) attains its extreme value, then $y(x)$ satisfies the following equation*

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0, \tag{5.2.2}$$

which is called the Euler's equation.

Proof In order to evidentiate that the value of the functional I computed at $y(x)$ is the extremum, we consider a vicinity of first order of the function $y(x)$

$$\{y(x)+\varepsilon\eta(x)\}_{\varepsilon},\ \eta(x) \in C^1[a,b],\ \eta(a)=\eta(b)=0.$$

The last conditions imposed to $\eta(x)$ mean that any function from this vicinity has the same ends like $y(x)$.

Let us compute the value of the functional I at an arbitrary representative of this vicinity:

$$I(y+\varepsilon\eta) = \int\limits_a^b L(y(x)+\varepsilon\eta(x), y'(x)+\varepsilon\eta'(x). \tag{5.2.3}$$

The integral in the right-hand member is a function of ε, that is

$$I(y+\varepsilon\eta) = F(\varepsilon).$$

But, for $\varepsilon = 0$ we obtain $I(y)$ and this value has been supposed to be the extreme value of the functional I. This means that $\varepsilon = 0$ satisfies the necessary condition for the extremum of the function F, that is

$$\left.\frac{dF(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \left[\frac{\partial L}{\partial (y + \varepsilon \eta)} \eta(x) + \frac{\partial L}{\partial (y' + \varepsilon \eta')} \eta'(x) \right] \mathrm{d}x \Bigg|_{\varepsilon=0} = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \left[\frac{\partial L}{\partial y} \eta(x) + \frac{\partial L}{\partial y'} \eta'(x) \right] \mathrm{d}x = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \frac{\partial L}{\partial y} \eta(x) \mathrm{d}x + \int_a^b \frac{\partial L}{\partial y'} \eta'(x) = 0. \tag{5.2.4}$$

We compute by parts the last integral

$$\int_a^b \frac{\partial L}{\partial y'} \eta'(x) = \frac{\partial L}{\partial y'} \eta(x) \Bigg|_a^b - \int_a^b \frac{d}{\mathrm{d}x} \left(\frac{\partial L}{\partial y} \right) \eta(x) \mathrm{d}x.$$

Due to the conditions $\eta(a) = \eta(b) = 0$ we obtain

$$\int_a^b \frac{\partial L}{\partial y'} \eta'(x) = - \int_a^b \frac{d}{\mathrm{d}x} \left(\frac{\partial L}{\partial y} \right) \eta(x) \mathrm{d}x,$$

such that from Eq. (5.2.3) it follows

$$\int_a^b \frac{\partial L}{\partial y} \eta(x) \mathrm{d}x - \int_a^b \frac{d}{\mathrm{d}x} \left(\frac{\partial L}{\partial y} \right) \eta(x) \mathrm{d}x = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \left[\frac{\partial L}{\partial y} \eta(x) - \frac{d}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) \right] \eta(x) \mathrm{d}x = 0.$$

Now, we can use the fundamental lemma whence it follows

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) = 0,$$

that is, we obtained Eq. (5.2.2) and the theorem is demonstrated. ■

Remark 5.1. Because $L = L(x, y(x), y'(x))$, we have

$$\frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial^2 L}{\partial y' \partial x} + \frac{\partial^2 L}{\partial y' \partial y} y' + \frac{\partial^2 L}{\partial y'^2} y'',$$

and the Euler's equation becomes

$$\frac{\partial L}{\partial y} - \frac{\partial^2 L}{\partial y' \partial x} - \frac{\partial^2 L}{\partial y' \partial y} y' - \frac{\partial^2 L}{\partial y'^2} y'' = 0.$$

So, we can see that the Euler's equation is an ordinary differential equation of second order, such that its general solution depends on two arbitrary real constants. These constants will be eliminated by using the conditions $y(a) = y_a$ and $y(b) = y_b$, knowing the fact that the numbers y_a and y_b are prescribed.

Remark 5.2. We must outline that not any solution of the Euler's equation is the extremum value of the functional I. The Euler's equation assures the contrary result: if the function is an extreme value of the functional I, then this function satisfies the Euler's equation. As in the classical theory of functions, any solution of the Euler's equation will be called a *stationary point* of the functional I.

In order to obtain an effective extremum of the functional I we will give a sufficient condition, as in the classical theory of functions.

In the following two propositions we obtain two prime integrals for the Euler's equation.

Proposition 5.2.1 *If the Lagrangean of the functional I does not depend on the function, then the Euler's equation admits the following prime integral*

$$\frac{\partial L}{\partial y'} = C, \tag{5.2.5}$$

where C is an arbitrary constant.

Proof Since the Lagrangean of the functional I does not depend on the function, we have

$$L = L(x, y'(x)) \Rightarrow \frac{\partial L}{\partial y} = 0,$$

such that the Euler's equation reduces to

$$\frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0 \Rightarrow \frac{\partial L}{\partial y'} = C = \text{constant},$$

and the proof of the proposition is concluded. ■

Proposition 5.2.2 *If the Lagrangean of the functional I does not explicitly depend on the variable x, then the Euler's equation admits the following prime integral*

$$L - y'\frac{\partial L}{\partial y'} = C, \tag{5.2.6}$$

where C is an arbitrary constant.

Proof Since the Lagrangean of the functional I does not explicitly depend of variable x, we have

$$L = L(y, y'(x)) \Rightarrow \frac{\partial L}{\partial x} = 0.$$

We will prove that the total derivative of the expression

$$L - y'\frac{\partial L}{\partial y'}$$

is null such that this expression will be a constant. By direct calculations, we have

$$\frac{d}{\mathrm{d}x}\left(L - y'\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}y' + \frac{\partial L}{\partial y'}y'' - y''\frac{\partial L}{\partial y'} - y'\frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) =$$

$$= y'\left[\frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y}\right],$$

such that, by using the Euler's equation, we get

$$\frac{d}{\mathrm{d}x}\left(L - y'\frac{\partial L}{\partial y'}\right) = 0,$$

whence it follows

$$L - y'\frac{\partial L}{\partial y'} = C = \text{constant},$$

and the proof of the proposition is concluded. ■

Remark. Let us consider that the Lagrangean is a linear function of the derivative of the unknown function, that is

$$L\left(x, y(x), y'(x)\right) = A(x, y)y'(x) + B(x, y),$$

where the functions $A(x, y)$ and $B(x, y)$ satisfy the condition

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}.$$

Then we obtain

$$\frac{\partial L}{\partial y} = \frac{\partial A}{\partial y}y' + \frac{\partial B}{\partial y},\ \frac{\partial L}{\partial y'} = A,$$

such that the Euler's equation becomes

$$\frac{\partial A}{\partial y} y' + \frac{\partial B}{\partial y} - \frac{d}{\mathrm{d}x}(A) = 0 \Rightarrow$$

$$\frac{\partial A}{\partial y} y' + \frac{\partial B}{\partial y} - \frac{\partial A}{\partial x} - \frac{\partial A}{\partial y} y' \equiv 0.$$

With other words, the Euler's equation becomes an identity and it does not have any solution. This fact can be explained as follows. The functional with the above Lagrangean has the form

$$I(y) = \int_a^b \left[A(x, y) \frac{\mathrm{d}y}{\mathrm{d}x} + B(x, y) \right] \mathrm{d}x =$$

$$= \int_{(a, y(a))}^{(b, y(b))} [A(x, y)\mathrm{d}y + B(x, y)\mathrm{d}x]\, \mathrm{d}x.$$

Because of the condition

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}$$

the above integral depends solely on the initial point $(a, y(a))$ and terminal point $(b, y(b))$ and is independent of the shape of the curve.

Application 1. Let us solve the problem of the brahistocrone curve. Consider the explicit form $y = y(x)$ for the curve that connects the points $O(0, 0)$ and $A(x_1, y_1)$. We know the classical Newton's law for a moving point (with known notations):

$$m\ddot{\mathbf{r}} = \mathbf{F},\ \mathbf{F} = m\mathbf{g} \Rightarrow m\ddot{\mathbf{r}}\dot{\mathbf{r}} = \mathbf{F}\dot{\mathbf{r}} \Rightarrow m\dot{\mathbf{v}}\mathbf{v} = \mathbf{F}\dot{\mathbf{r}} \Rightarrow$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\mathbf{v}^2}{2}\right) = \mathbf{F}\dot{\mathbf{r}} \Rightarrow \mathrm{d}\left(\frac{m\mathbf{v}^2}{2}\right) = \mathbf{F}\dot{\mathbf{r}} = \mathbf{F}\mathrm{d}\mathbf{r}.$$

But $\mathrm{d}\mathbf{r} = \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j}$ such that the last equation becomes

$$\mathrm{d}\left(\frac{m\mathbf{v}^2}{2}\right) = g\mathrm{d}y = \mathrm{d}(mgy) \Rightarrow \frac{m\mathbf{v}^2}{2} = mgy \Rightarrow v^2 = 2gy \Rightarrow v = \sqrt{2gy}.$$

On the other hand, we have

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \Rightarrow \mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}},$$

after we used that the initial speed is null and the element of arc on the curve $y = y(x)$ has the formula $ds = \sqrt{1 + y'^2}$.

We integrate the last equation allong the curve $y = y(x)$ between the points O and A and obtain

$$t = \int_0^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} \mathrm{d}x.$$

Of course, considering another curve between O and A we will obtain another value of the time t, such that we must write

$$t(y) = \frac{1}{\sqrt{2g}} \int_0^{x_1} \sqrt{\frac{1 + y'^2}{y}} \mathrm{d}x.$$

So, we must find the function $y(x)$ for which the functional $t(y)$ has the minimum value. It is easy to see that

$$L = \sqrt{\frac{1 + y'^2}{y}}.$$

Because $L = L(y(x), y'(x)$, that is the Lagrangean does not depend explicitely on x, we can use the prime integral, proved in the Proposition 5.2.2:

$$L - y' \frac{\partial L}{\partial y'} = C.$$

So, we obtain the equation

$$\sqrt{\frac{1 + y'^2}{y}} - y' \frac{1}{\sqrt{y}} \frac{y'}{\sqrt{1 + y'^2}} = C \Rightarrow \frac{1 + y'^2 - y'^2}{\sqrt{y(1 + y'^2)}} = C \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{y(1 + y'^2)}} = C \Rightarrow y\left(1 + y'^2\right) = \frac{1}{C^2} = C_1.$$

The last equation is a parametric equation. If we denote $y' = \cot u$, it results

$$1 + y'^2 = 1 + \cot^2 u = \frac{1}{\sin^2 u} \Rightarrow$$

$$\Rightarrow y(u) = C_1 \sin^2 u = \frac{C_1}{2}(1 - \cos 2u). \tag{5.2.7}$$

On the other hand, we have

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{1}{y'} = \frac{\mathrm{d}y}{\mathrm{d}u}\tan u,$$

such that

$$\frac{\mathrm{d}x}{\mathrm{d}u} = C_1 \sin 2u \frac{\sin u}{\cos u} = 2C_1 \sin^2 u = C_1(1 - \cos 2u).$$

After a simple integration, we find

$$x(u) = C_1(u - \frac{\sin 2u}{2} = \frac{C_1}{2}(2u - \sin 2u) + C_2.$$

Taking into account this relation together with Eq. (5.2.7), we have the parametric form of the brahistocrone curve

$$\begin{cases} x(u) = \frac{C_1}{2}(2u - \sin 2u) + C_2 \\ y(u) = C_1 \sin^2 u = \frac{C_1}{2}(1 - \cos 2u). \end{cases} \tag{5.2.8}$$

The constants C_1 and C_2 will be determined using the condition that the curve passes by the points $O(0, 0)$ and $A(x_1, y_1)$.

In Eq. (5.2.8) we have a family of cycloides.

Application 2. Let us find the point of extreme for the following functional

$$I(y) = \int_0^{\ln 2} \left(e^{-x}y'^2 - e^x y^2\right)\mathrm{d}x,\ \ y(0) = a,\ \ y(\ln 2) = b,$$

where a and b are prescribed real numbers.

Using the Euler's equation, we obtain

$$-2e^x y - \frac{d}{\mathrm{d}x}\left(2e^{-x}y'\right) = 0 \Rightarrow \frac{d}{\mathrm{d}x}\left(2e^{-x}y'\right) + e^x y = 0,$$

after we simplified by (-2). We use the rule of the derivation for the product and multiply by e^x:

$$-e^{-x}y' + e^{-x}y'' + e^x y = 0\big| .e^x \Rightarrow y'' - y' + e^{2x}y = 0.$$

Using the change of variable $e^x = t$ we obtain

$$e^x \mathrm{d}x = \mathrm{d}t \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = e^x = t \Rightarrow y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \dot{y}t \Rightarrow$$

$$\Rightarrow y'' = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{d}{\mathrm{d}t}(\dot{y}t) = t\left(\ddot{y}t + \dot{y}\right),$$

such that the Euler's equation becomes

$$t^2\ddot{y} + t\dot{y} - t\dot{y} + t^y = 0 \Rightarrow \ddot{y} + y = 0.$$

Since the characteristic equation $r^2 + 1 = 0$ has the roots $\pm i$ we obtain the solution

$$y(t) = C_1 \cos t + C_2 \sin t \Rightarrow y(x) = C_1 \cos e^x + C_2 \sin e^x,$$

where the constants C_1 and C_2 can be determined by using the initial conditions.

5.3 Generalizations of Euler's Equation

In this paragraph we will prove a few generalizations of the Euler's equation, with regard to a number of functions which appear under the Lagrangean, the order of the derivative of the function $y(x)$ which appears under the Lagrangean of the functional and with regard to the number of variables of the unkown function.

In the first following theorem, called the *Lagrange-Euler's system of equations*, we extend the fundamental result of Euler to the case where the functional (also, the Lagrangean) depends of many unknown functions. Let us consider that the functional depends on n unknown functions, each depends on a single variable x, that is,

$$I(y_1, y_2, \ldots, y_n) = \int_a^b L\left(x, y_1(x), y_2(x), \ldots, y_n(x), y_1'(x), y_2'(x), \ldots, y_n'(x)\right) \mathrm{d}x,$$

where $y_i = y_i(x),\ i = 1, 2, \ldots, n.$

Theorem 5.3.1 *Suppose that the functional I attains its extreme value in the functions $(y_1, y_2, \ldots, y_n)$. Then these functions satisfy the following system of equations*

$$\frac{\partial L}{\partial y_1} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y_1'}\right) = 0,$$

$$\frac{\partial L}{\partial y_2} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y_2'}\right) = 0$$

$$-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -$$

$$\frac{\partial L}{\partial y_n} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y_n'}\right) = 0. \tag{5.3.1}$$

Proof Without loss of generality, we suppose that $n = 2$. Therefore, consider the functional

$$I(y, z) = \int_a^b L\left(x, y(x), z(x), y'(x), z'(x)\right) \mathrm{d}x,$$

where $y = y(x)$ and $z = z(x)$.

Supposing that the functional attains its extreme value in the functions $(y(x), z(x))$, we must prove that these functions satisfy the following system of equations

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0,$$

$$\frac{\partial L}{\partial z} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial z'}\right) = 0.$$

Consider two vicinities of the functions $y(x)$ and $z(x)$, respectively,

$$\{y(x) + \varepsilon_1\eta_1(x)\}_{\varepsilon_1}, \{z(x) + \varepsilon_2\eta_2(x)\}_{\varepsilon_2},$$

where $\eta_1(x)$ and $\eta_2(x)$ are functions of class C^1 on the interval $[a, b]$ and satisfy the conditions

$$\eta_1(a) = \eta_1(b) = 0, \ \eta_2(a) = \eta_2(b) = 0.$$

We will compute the value of the functional I for an arbitrary representative of each vicinity:

$$I\left(y + \varepsilon_1\eta_1, z + \varepsilon_2\eta_2\right) =$$

$$= \int_a^b L\left(x, y(x) + \varepsilon_1\eta_1(x), z(x) + \varepsilon_2\eta_2(x), y'(x) + \varepsilon_1\eta_1'(x), z'(x) + \varepsilon_2\eta_2'(x)\right) \mathrm{d}x.$$

Of course, we can write

$$I\left(y + \varepsilon_1\eta_1, z + \varepsilon_2\eta_2\right) = I\left(\varepsilon_1, \varepsilon_2\right).$$

For $\varepsilon_1 = \varepsilon_2 = 0$ the function $y(x) + \varepsilon_1\eta_1(x)$ from first vicinity becomes $y(x)$ and the function $z(x) + \varepsilon_2\eta_2(x)$ from second vicinity becomes $z(x)$ and the functional I has its extreme value for the functions $y(x)$ and $z(x)$. So, we can conclude that the point $(0, 0)$ is the point of the extreme for the function $I(\varepsilon_1, \varepsilon_2)$ and must satisfy the necessary condition of extreme, that is

$$\left.\frac{\partial I(\varepsilon_1, \varepsilon_2)}{\partial\varepsilon_1}\right|_{\varepsilon_1=0} = 0, \ \left.\frac{\partial I(\varepsilon_1, \varepsilon_2)}{\partial\varepsilon_2}\right|_{\varepsilon_2=0} = 0.$$

Now, we derivate with respect to ε_1 and ε_2 the integral that defines the function $I(\varepsilon_1, \varepsilon_2)$:

$$\int_a^b \left[\frac{\partial L}{\partial(y+\varepsilon_1\eta_1}\eta_1(x) + \frac{\partial L}{\partial(y'+\varepsilon_1\eta_1'}\eta_1'(x)\right]\mathrm{d}x\Bigg|_{\varepsilon_1=0} = 0,$$

$$\int_a^b \left[\frac{\partial L}{\partial(y+\varepsilon_2\eta_2}\eta_2(x) + \frac{\partial L}{\partial(y'+\varepsilon_2\eta_2'}\eta_2'(x)\right]\mathrm{d}x\Bigg|_{\varepsilon_2=0} = 0.$$

Taking into account that $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$, we obtain

$$\int_a^b \left[\frac{\partial L}{\partial y}\eta_1(x) + \frac{\partial L}{\partial y'}\eta_1'(x)\right]\mathrm{d}x = 0 \Rightarrow \int_a^b \frac{\partial L}{\partial y}\eta_1(x)\mathrm{d}x + \int_a^b \frac{\partial L}{\partial y'}\eta_1'(x)\mathrm{d}x = 0,$$

and

$$\int_a^b \left[\frac{\partial L}{\partial z}\eta_2(x) + \frac{\partial L}{\partial z'}\eta_2'(x)\right]\mathrm{d}x = 0 \Rightarrow \int_a^b \frac{\partial L}{\partial z}\eta_2(x)\mathrm{d}x + \int_a^b \frac{\partial L}{\partial z'}\eta_2'(x)\mathrm{d}x = 0.$$

We integrate by parts the last integrals of the above relations. Thus,

$$\int_a^b \frac{\partial L}{\partial y'}\eta_1'(x)\mathrm{d}x = \frac{\partial L}{\partial y'}\eta_1\Bigg|_a^b - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta_1(x)\mathrm{d}x$$

and

$$\int_a^b \frac{\partial L}{\partial z'}\eta_2'(x)\mathrm{d}x = \frac{\partial L}{\partial z'}\eta_2\Bigg|_a^b - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial z'}\right)\eta_2(x)\mathrm{d}x.$$

But, by hypothesis, we have

$$\eta_1(a) = \eta_1(b) = 0, \;\; \eta_2(a) = \eta_2(b) = 0,$$

and then the integrals become

$$\int_a^b \frac{\partial L}{\partial y'}\eta_1'(x)\mathrm{d}x = -\int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta_1(x)\mathrm{d}x$$

$$\int_a^b \frac{\partial L}{\partial z'}\eta_2'(x)\mathrm{d}x = -\int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial z'}\right)\eta_2(x)\mathrm{d}x.$$

Finally, the conditions of extreme become

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\eta_1(x)\mathrm{d}x = 0,$$

and, respectively,

$$\int_a^b \left[\frac{\partial L}{\partial z} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial z'}\right)\right]\eta_2(x)\mathrm{d}x = 0.$$

For both integrals in the left-hand member of these relations, we can use the fundamental lemma whence it follows

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0,$$

$$\frac{\partial L}{\partial z} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial z'}\right) = 0$$

and the theorem is concluded. ■

Application 1. Let us find the functions $y(x)$ and $z(x)$ that make the following integral extremal

$$I(y,z) = \int_0^\pi \left(2yz - 2y^2 + y'^2 - z'^2\right)\mathrm{d}x,\ \ y(0) = 0,\ y(\pi) = 1,\ z(0) = 0,\ z(\pi) = 1.$$

Taking into account that the Lagrangean is

$$L\left(x, y(x), z(x), y'(x), z'(x)\right) = \left(2yz - 2y^2 + y'^2 - z'^2\right),$$

the Euler-Lagrange's system of equations becomes

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0,\ \ \frac{\partial L}{\partial z} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial z'}\right) = 0 \Rightarrow$$

$$2z - 4y - \frac{d}{\mathrm{d}x}\left(2y'\right) = 0,\ \ 2y - \frac{d}{\mathrm{d}x}\left(-2z'\right) = 0, \Rightarrow$$

$$y'' + 2y - z = 0,\ \ z'' + y = 0, \Rightarrow$$

$$y^{(4)} + 2y'' - z'' = 0,\ z'' = -y \Rightarrow y^{(4)} + 2y'' + y = 0.$$

Since the characteristic equation

$$r^4 + 2r^2 + 1 = 0,$$

has the double complex conjugated roots $\pm i$, our differential equation has the general solution equal to

$$y(x) = (Ax + B)\cos x + (Cx + D)\sin x.$$

Using the initial conditions $y(0) = 0$ and $y(\pi) = 1$, we find $B = 0$ and $A = -1/\pi$. Then the function $z(x)$ can be obtained by using the equation $z(x) = y''(x) + 2y(x)$ and the initial conditions $z(0) = 0$, $z(\pi) = 1$. Finally, we obtain the solutions

$$y(x) = -\frac{x}{\pi}\cos x + D\sin x$$

$$z(x) = \frac{1}{\pi}(2\sin x - \pi\cos x) + D\sin x,$$

where D is an arbitrary constant.

Another generalization of the Euler's equation can be obtained considering the case when the functional I depends on only one function y, but the Lagrangean of the functional depends on superior derivatives of the function y. The result is due to Poisson and Euler and is proved in the following theorem. Based on the above considerations, we will consider the functional I in the form:

$$I(y) = \int_a^b L\left(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)\right) \mathrm{d}x.$$

Theorem 5.3.2 *If the function $y(x)$ is the function where the functional I, for which the Lagrangean depends on superior derivatives of $y(x)$, attains its extreme value, then $y(x)$ satisfies the following equation*

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) + \frac{d^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right) + \cdots + (-1)^n \frac{d^n}{\mathrm{d}x^n}\left(\frac{\partial L}{\partial y^{(n)}}\right) = 0,$$

called the Poisson-Euler's equation.

Proof For the sake of simplicity of calculations, but without loss of generality, we consider only the case $n = 2$. Consequently, the Lagrangean depends, more, on $y''(x)$, that is,

$$I(y) = \int_a^b L\left(x, y(x), y'(x), y''(x)\right) \mathrm{d}x.$$

Let us prove that the function $y(x)$ satisfies the equation

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) + \frac{d^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right) = 0.$$

Together with function $y(x)$ consider a vicinity of order two that contains functions of the form

$$\{y(x) + \varepsilon\eta(x)\}_\varepsilon\,,$$

where ε is a small parameter and $\eta(x)$ is a function of class C^2 on the interval $[a, b]$ satisfying the conditions

$$\eta(a) = \eta(b) = 0,\ \ \textit{and}\ \eta'(a) = \eta'(b) = 0.$$

In order to evidentiate that the value of the functional I is extreme for the function $y(x)$, we compute the value of I for an arbitrary representative of the above vicinity:

$$I(y + \varepsilon\eta) = \int_a^b L\left(x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x), y''(x) + \varepsilon\eta''(x)\right) \mathrm{d}x.$$

So, we obtain

$$I(y + \varepsilon\eta) = I(\varepsilon).$$

But for $\varepsilon = 0$ the representative of the vicinity becomes even $y(x)$ and the functional I attains its extreme value at $y(x)$. So, we conclude that $\varepsilon = 0$ is the point of extreme for function $I(\varepsilon)$ and then, it must satisfy the necessary condition of extreme, that is

$$\left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

We introduce the derivative under the integral because the ends of the integral do not depend on ε:

$$\left.\int_a^b \left[\frac{\partial L}{\partial(y + \varepsilon\eta)}\eta(x) + \frac{\partial L}{\partial(y' + \varepsilon\eta')}\eta'(x) + \frac{\partial L}{\partial(y'' + \varepsilon\eta'')}\eta''(x)\right] \mathrm{d}x\right|_{\varepsilon=0} = 0.$$

Using the fact that $\varepsilon = 0$, we obtain

$$\int_a^b \left[\frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x) + \frac{\partial L}{\partial y''}\eta''(x)\right] \mathrm{d}x = 0.$$

Now, we decompose the integral from the left-hand member in three integrals and compute the last two by parts:

$$\int_a^b \frac{\partial L}{\partial y}\eta(x)\mathrm{d}x + \int_a^b \frac{\partial L}{\partial y'}\eta'(x)\mathrm{d}x + \int_a^b \frac{\partial L}{\partial y''}\eta''(x)\mathrm{d}x = I_1 + I_2 + I_3 = 0. \quad (5.3.2)$$

Integrating by parts the second integral from Eq. (5.3.2), I_2, we obtain

$$I_2 = \int_a^b \frac{\partial L}{\partial y'}\eta'(x)\mathrm{d}x = \left.\frac{\partial L}{\partial y'}\eta\right|_a^b - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x.$$

Taking into account that $\eta(a) = \eta(b) = 0$, it results

$$I_2 = \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x.$$

Now, we integrate by parts the third integral from Eq. (5.3.2):

$$I_3 = \int_a^b \frac{\partial L}{\partial y''}\eta''(x)\mathrm{d}x = \left.\frac{\partial L}{\partial y''}\eta'\right|_a^b - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y''}\right)\eta'(x)\mathrm{d}x.$$

Taking into account that $\eta'(a) = \eta'(b) = 0$, it results

$$I_3 = \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y''}\right)\eta'(x)\mathrm{d}x.$$

Once again, we integrate also by parts

$$I_3 = -\left.\frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y''}\right)\eta\right|_a^b + \int_a^b \frac{d^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right)\eta(x)\mathrm{d}x.$$

But, by hypothesis, we have $\eta(a) = \eta(b) = 0$, and then the integral I_3 becomes:

$$I_3 = \int_a^b \frac{d^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right)\eta(x)\mathrm{d}x.$$

With the founded form for I_2 and I_3, the relation (5.3.2) becomes

$$\int_a^b \frac{\partial L}{\partial y}\eta(x)\mathrm{d}x - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x + \int_a^b \frac{d^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right)\eta(x)\mathrm{d}x = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x) + \frac{d^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right)\right]\eta(x)\mathrm{d}x = 0.$$

Now, we can apply the fundamental lemma and the previous relation leads to the equation

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) + \frac{d^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right) = 0,$$

which is the Poisson-Euler's equation and the theorem is proved. ∎

Application 1. As an application of the Poisson-Euler's equation we will find the function that makes the following functional extremal

$$I(y) = \int_1^e \left(x^2y'' - 2x^2y'^2\right)\mathrm{d}x,\ \ y(1) = 0,\ \ y(e) = 1.$$

Taking into account that the Lagrangean of the functional is

$$L\left(x, y(x), y'(x), y''(x)\right) = x^2y'' - 2x^2y'^2,$$

we obtain

$$\frac{\partial L}{\partial y} = 0,\ \frac{\partial L}{\partial y'} = -4x^2y',\ \frac{\partial L}{\partial y''} = x^2,$$

and then the Poisson-Euler's equation reduces to

$$-\frac{d}{\mathrm{d}x}\left(-4x^2y'\right) + \frac{d^2}{\mathrm{d}x^2}\left(x^2\right) = 0 \Rightarrow 4\left(4x^2y'' + 2xy'\right) + 2 = 0 \Rightarrow$$

$$\Rightarrow 2x^2y'' + 4xy' + 1 = 0.$$

If we denote by $y'(x) = z(x)$, the previous equation reduces to

$$2x^2z' + 4xz + 1 = 0 \Rightarrow z' + \frac{2}{x}z = -\frac{1}{2x^2}.$$

The last equation is an ordinary linear differential equation of first order such that its solution has the form

$$z(x) = e^{-2\ln x}\left(C - \int \frac{1}{2x^2}e^{2\ln x}\mathrm{d}x\right) = \frac{1}{x^2}\left(C - \int \frac{1}{2x^2}x^2\mathrm{d}x\right) = \frac{C}{x^2} - \frac{1}{2x},$$

where C is an arbitrary constant. Taking into account the above notation, with regard to the function $y(x)$ we obtain

$$y(x) = -\frac{C}{x} - \frac{1}{2}\ln x + C_1.$$

In order to obtain the constants C and C_1 we will use the initial conditions $y(1) = 0$ and $y(e) = 1$ such that

$$C = C_1 = \frac{3e}{2(e-1)}.$$

Therefore, the function that makes the given functional extremal is

$$y(x) = \frac{3e}{2(e-1)}\left(1 - \frac{1}{x}\right) - \frac{1}{2}\ln x.$$

The last generalization of the Euler's equation that we will obtain is regarding to the number of the independent variables. We will denote by u the unknown function and suppose that it is dependent on n independent variables, that is,

$$u = u(x_1, x_2, \ldots, x_n).$$

Also, we suppose that the functional depends only on one unknown function, u, and the Lagrangean is dependent only on the function u and its first partial derivative:

$$I(u) = \int_\Omega L\left(x, u(x), \frac{\partial u}{\partial x_i}\right)\mathrm{d}x,$$

where x is a vectorial variable $x = (x_1, x_2, \ldots, x_n)$ and Ω is a domain in a n-dimensional space, as such, we have a multiple integral.

For the sake of simplicity of calculations, but without loss the generality, we consider only the case $n = 2$. Consequently, the unknown function u depends only on two variables, namely, x and y, the Lagrangean depends on the unknown function u and on the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$. As such, we have the following functional

$$I(u) = \iint\limits_{\Omega} L\left(x, y, u(x, y), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \mathrm{d}x\mathrm{d}y.$$

The following theorem, due to Ostrogradski and Euler, gives the equation verified by the function u that makes the above functional extremal.

Let Ω be a bounded domain in the two-dimensional space R^2 having the smooth boundary $\Gamma = \partial\Omega$. Consider that the unknown function u and the Lagrangean L are regular functions, $u,\ L \in C^1(\Omega)$ and use the well known Monge's notations

$$u_x = \frac{\partial u}{\partial x},\ u_y = \frac{\partial u}{\partial y}.$$

Consequently, we must make the following functional extremal

$$I(u) = \iint\limits_{\Omega} L\left(x, y, u(x, y), u_x, u_y\right) \mathrm{d}x\mathrm{d}y. \tag{5.3.3}$$

Theorem 5.3.3 *If the function u extremates the functional I from Eq. (5.3.3) then it satisfies the following equation*

$$\frac{\partial L}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right) = 0,$$

which is called the Ostrogradski-Euler's equation.

Proof Together with the function u we consider a vicinity of first order containing functions of the form

$$\{u(x, y) + \varepsilon\eta(x, y)\}_{\varepsilon},$$

where ε is a small arbitrary parameter and $\eta(x, y)$ is a function of the class $\eta(x, y) \in C^1(\Omega)$ satisfying the condition

$$\eta(x, y) = 0,\ \forall(x, y) \in \Gamma = \partial\Omega. \tag{5.3.4}$$

The last conditions means that every function from the vicinity has the same ends like $u(x, u)$.

Let us compute the value of the functional I for an arbitrary representative of this vicinity.

$$I(u+\varepsilon\eta) = \iint\limits_{\Omega} L\big(x, y, u(x, y)+\varepsilon\eta(x, y), u_x(x, y)+\varepsilon\eta_x(x, y), u_y(x, y)+\varepsilon\eta_y(x, y)\big)\, \mathrm{d}x\mathrm{d}y.$$

We obtain a function that depends only on ε, $I(\varepsilon)$. For $\varepsilon = 0$ the function from the vicinity, $u(x, y) + \varepsilon\eta(x, y)$ reduces to the function $u(x, y)$ which is the point of the

extreme value for the functional I such that we deduce that $\varepsilon = 0$ must satisfy the necessary condition for the extremum, that is

$$\left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

Introducing the derivative under the integral, it follows

$$\left.\iint\limits_{\Omega}\left[\frac{\partial L}{\partial(u+\varepsilon\eta)}\eta(x, y)+\frac{\partial L}{\partial(u_x+\varepsilon\eta_x)}\eta_x(x, y)+\frac{\partial L}{\partial(u_y+\varepsilon\eta_y)}\eta_y(x, y)\right]\mathrm{d}x\mathrm{d}y\right|_{\varepsilon=0}=0.$$

If we take into account the fact that $\varepsilon = 0$, the previous relation becomes

$$\iint\limits_{\Omega}\left[\frac{\partial L}{\partial u}\eta(x, y)+\frac{\partial L}{\partial u_x}\eta_x(x, y)+\frac{\partial L}{\partial u_y}\eta_y(x, y)\right]\mathrm{d}x\mathrm{d}y = 0. \tag{5.3.5}$$

Now, we decompose the integral in three integrals and compute by parts the last two

$$\iint\limits_{\Omega}\frac{\partial L}{\partial u}\eta(x, y)\mathrm{d}x\mathrm{d}y+\iint\limits_{\Omega}\frac{\partial L}{\partial u_x}\eta_x(x, y)\mathrm{d}x\mathrm{d}y+\iint\limits_{\Omega}\frac{\partial L}{\partial u_y}\eta_y(x, y)\mathrm{d}x\mathrm{d}y =$$

$$= I_1 + I_2 + I_3 = 0.$$

With regard to I_2 we have the following estimations

$$I_2 = \iint\limits_{\Omega}\frac{\partial L}{\partial u_x}\eta_x(x, y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\eta\right)-\eta\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right)\right]\mathrm{d}x\mathrm{d}y$$

and then we can write

$$I_2 = \iint\limits_{\Omega}\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\eta\right)\mathrm{d}x\mathrm{d}y-\iint\limits_{\Omega}\eta\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right)\mathrm{d}x\mathrm{d}y. \tag{5.3.6}$$

Also, with regard to I_3 we have the following estimations

$$I_3 = \iint\limits_{\Omega}\frac{\partial L}{\partial u_y}\eta_y(x, y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega}\left[\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\eta\right)-\eta\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right)\right]\mathrm{d}x\mathrm{d}y$$

and then we can write

$$I_3 = \iint\limits_{\Omega} \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\eta\right)\mathrm{d}x\mathrm{d}y - \iint\limits_{\Omega} \eta\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right)\mathrm{d}x\mathrm{d}y. \tag{5.3.7}$$

We remember now the well known Green's formula, with regard to the connection between the line integral and double integral. So, if Ω is a bounded domain in the space R^2 with the boundary Γ which is a closed smooth curve, then the Green's formula asserts the following connection

$$\oint\limits_{\Gamma} P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = \iint\limits_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\right)\mathrm{d}x\mathrm{d}y.$$

With this formula in mind, we add, member by member, the relations (5.3.6) and (5.3.7)

$$I_2+I_3=\iint\limits_{\Omega}\left(\frac{\partial L}{\partial u_x}\eta_x+\frac{\partial L}{\partial u_y}\eta_y\right)\mathrm{d}x\mathrm{d}y=\iint\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(\eta\frac{\partial L}{\partial u_x}\right)+\frac{\partial}{\partial y}\left(\eta\frac{\partial L}{\partial u_y}\right)\right]\mathrm{d}x\mathrm{d}y-$$

$$-\iint\limits_{\Omega}\left[\eta\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right)+\eta\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right)\right]\mathrm{d}x\mathrm{d}y.$$

On the first integral in the right-hand member of the previous relation we apply the Green's formula

$$\iint\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(\eta\frac{\partial L}{\partial u_x}\right)+\frac{\partial}{\partial y}\left(\eta\frac{\partial L}{\partial u_y}\right)\right]\mathrm{d}x\mathrm{d}y = \oint\limits_{\Gamma} -\eta\frac{\partial L}{\partial u_y}\mathrm{d}x + \eta\frac{\partial L}{\partial u_x}\mathrm{d}y = 0,$$

where we used the condition (5.3.4). In this manner, the sum $I_2 + I_3$ reduces to

$$I_2 + I_3 = \iint\limits_{\Omega}\left[\eta\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right)+\eta\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right)\right]\mathrm{d}x\mathrm{d}y.$$

Introducing this results in Eq. (5.3.5), we obtain

$$\iint\limits_{\Omega}\left[\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right)\right]\eta(x, y)\mathrm{d}x\mathrm{d}y = 0.$$

Now, we can use the fundamental lemma such that the previous relation leads to

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right) = 0,$$

which is the Ostrogradski-Euler's equation, that concludes the proof is concluded. ■

Application 1. Let us use the Ostrogradski-Euler's equation to determine the function that makes the following functional extremal

$$I(u) = \iint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] \mathrm{d}x\mathrm{d}y,$$

where $u = u(x, y),\ u \in C^2(\Omega)$ and Ω is bounded domain in the Euclidean space R^2. The Lagrangean of our functional is

$$L\left(x, y, u(x, y), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 .$$

Then

$$\frac{\partial L}{\partial u} = 0,\ \frac{\partial L}{\partial u_x} = 2\frac{\partial u}{\partial x} = u_x,\ \frac{\partial L}{\partial u_y} = 2\frac{\partial u}{\partial y} = u_y,$$

such that the Ostrogradski-Euler's equation becomes

$$\frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}\left(u_y\right) = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Thus, the function $u(x, y)$ satisfies the Laplace's equation and, as we know, is called a *harmonic function*. It is known that in the Euclidean space R^2 a harmonic function has the form

$$u(x, y) = C \ln \frac{1}{\sqrt{x^2 + y^2}},$$

where C is an arbitrary constant. In the Euclidean space R^n, $n \geq 3$ a harmonic function has the form

$$u(x) = C\frac{1}{r},$$

where x is a vectorial variable, $x = (x_1, x_2, \ldots, x_n)$, C is an arbitrary constant and r is the Euclidean distance, that is,

$$r = \sqrt{\sum_{k=1}^{n} x_k^2}$$

5.4 Sufficent Conditions for Extremum

As we alreay said, a function that satisfies the Euler's equation is not automatically an extremal value of the functional. This is only the necessary condition for extremum. By analogy with the classical theory of functions, to a function that satisfies the Euler's equation must impose supplementary conditions to be effective an extremum of the functional. In the case of the functionals these sufficient conditions will be obtained with the aid of variation of order two of the functional. Firstly, in the following theorem, we will obtain the form of variation of order two of a functional of integral type.

Theorem 5.4.1 *Let $y(x)$ be a function that satisfies the Euler's equation regarding the functional*

$$I(y) = \int_a^b L\left(x, y(x), y'(x)\right) \mathrm{d}x.$$

Consider a vicinity of the function $y(x)$ which consists of the functions

$$\{y(x) + \varepsilon\eta(x)\}_\varepsilon ,$$

where ε is a small parameter and the functions $\eta(x)$ satisfy the usual conditions $\eta \in C^1[a, b]$, $\eta(a) = 0$ and $\eta(a) = 0$. If the functional is computed for a function $y(x)\varepsilon\eta(x)$, belonging to the above vicinity, then its variation of order two has the expression

$$\left.\frac{\delta^2 I}{d\varepsilon^2}\right|_{\varepsilon=0} = \int_a^b \frac{\partial^2 L}{\partial y'^2}\eta'^2(x)\mathrm{d}x.$$

Proof By direct calculations, we obtain

$$\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0} = \frac{\delta^2}{\delta\varepsilon^2}\left[\int_a^b L\left(x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x)\right) \mathrm{d}x\right]_{\varepsilon=0} =$$

$$= \frac{\delta}{\delta\varepsilon}\left[\int_a^b \left(\frac{\partial L}{\partial(y + \varepsilon\eta)}\eta(x) + \frac{\partial L}{\partial(y' + \varepsilon\eta')}\eta'(x)\right) \mathrm{d}x\right]_{\varepsilon=0} =$$

$$= \int_a^b \left[\frac{\partial^2 L}{\partial(y+\varepsilon\eta)^2}\eta^2(x)+2\frac{\partial^2 L}{\partial(y+\varepsilon\eta)\partial(y'+\varepsilon\eta')}\eta(x)\eta'(x)+\frac{\partial^2 L}{\partial(y'+\varepsilon\eta')^2}\eta'^2(x)\right]\mathrm{d}x\Bigg|_{\varepsilon=0}$$

Taking into account that $\varepsilon = 0$, we obtain

$$\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0} = \int_a^b \left[\frac{\partial^2 L}{\partial y^2}\eta^2(x) + 2\frac{\partial^2 L}{\partial y \partial y'}\eta(x)\eta'(x) + \frac{\partial^2 L}{\partial y'^2}\eta'^2(x)\right] \mathrm{d}x,$$

such that the variation of order two the functional can be restated as follows

$$\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0} = \int_a^b \frac{\partial^2 L}{\partial y^2}\eta^2(x)\mathrm{d}x + 2\int_a^b \frac{\partial^2 L}{\partial y \partial y'}\eta(x)\eta'(x)\mathrm{d}x +$$

$$+\int_a^b \frac{\partial^2 L}{\partial y'^2}\eta'^2(x)\mathrm{d}x = I_1 + I_2 + I_3. \tag{5.4.1}$$

We integrate by parts the second integral from Eq. (5.4.1), that is, I_2:

$$\int_a^b \frac{\partial^2 L}{\partial y \partial y'}\eta(x)2\eta'(x)\mathrm{d}x = \left.\frac{\partial^2 L}{\partial y \partial y'}\eta^2(x)\right|_a^b - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y \partial y'}\right)\eta^2(x)\mathrm{d}x.$$

But $\eta(a) = \eta(b) = 0$ and then I_2 becomes

$$I_2 = -\int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y \partial y'}\right)\eta^2(x)\mathrm{d}x.$$

Introducing this form of I_2 in Eq. (5.4.1) it follows

$$\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0} = \int_a^b \left\{\left[\frac{\partial^2 L}{\partial y^2} - \frac{d}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y \partial y'}\right)\right]\eta^2(x) + \frac{\partial^2 L}{\partial y'^2}\eta'^2(x)\right\} \mathrm{d}x.$$

Let us prove that the first part of the last integral is equal to zero. Starting from the Euler's equation, by using the derivative with respect to y, it follows

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0\Big|_y' \Rightarrow$$

$$\Rightarrow \frac{\partial^2 L}{\partial y^2} - \frac{d}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y \partial y'}\right) = 0.$$

Finally, the second variation of the functional reduces to

$$\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0} = \int_a^b \frac{\partial^2 L}{\partial y'^2}\eta'^2(x)\mathrm{d}x,$$

such that the desired result is concluded. ■

Using the second variation of the functional, we will find a sufficient condition for the extremum.

In the following theorem we prove the sufficient condition that must satisfy the function $y(x)$, which verifies the Euler's equation, to be effectively, the point of minimum for the functional I. The result is due to Legendre.

Theorem 5.4.2 *If the second variation of the functional I is positive, for every function $\eta(x)$ such that $\eta(x) \in C^1[a,b]$, $\eta(a) = \eta(b) = 0$, then*

$$\frac{\partial^2 L}{\partial y'^2} \geq 0.$$

Proof According to the hypothesis, we have

$$\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0} \geq 0 \Rightarrow$$

$$\Rightarrow \int_a^b \frac{\partial^2 L}{\partial y'^2}\eta'^2(x)\mathrm{d}x \geq 0.$$

Suppose that there exists a point $x_0 \in [a,b]$ such that

$$\frac{\partial^2 L}{\partial y'^2} < 0.$$

Since $L \in C^2$, we deduce that

$$\frac{\partial^2 L}{\partial y'^2} \in C^0,$$

that is, it is a continuous function. Therefore,

$$\frac{\partial^2 L}{\partial y'^2}$$

is negative on the whole vicinity of the point x_0. Thus, we can write

$$\frac{\partial^2 L}{\partial y'^2} < -\frac{\alpha^2}{2}, \ \forall x \in [x_0-\varepsilon, x_0+\varepsilon],$$

where ε is a small parameter. The inequality from the hypothesis takes place for any function $\eta(x)$ that satisfies the above conditions. That means, the given inequality takes place for a particular $\eta(x)$, say $\eta_0(x)$ that we indicate as follows

$$\eta_0(x) = \begin{cases} 0, \text{ if } x \in [a, x_0 - \varepsilon), \\ \sin^2(x^2 - \varepsilon^2)/\varepsilon^2, \text{ if } x \in [x_0 - \varepsilon, x_0 + \varepsilon], \\ 0, \text{ if } x \in (x_0 + \varepsilon, b]. \end{cases}$$

It is easy to see that, by definition, $\eta_0(a) = \eta_0(b) = 0$.
By direct calculations, we obtain

$$\lim_{x \nearrow x_0 - \varepsilon} \eta_0(x) = \lim_{x \searrow x_0 - \varepsilon} \eta_0(x) = 0,$$

and, similarly,

$$\lim_{x \nearrow x_0 + \varepsilon} \eta_0(x) = \lim_{x \searrow x_0 + \varepsilon} \eta_0(x) = 0.$$

That is, the function $\eta_0(x)$ is continuous at the points $x_0 \pm \varepsilon$. Using the Lagrange's consequence, one can prove that the function $\eta_0(x)$ has derivatives at the points $x_0 \pm \varepsilon$, as follows

$$\lim_{x \nearrow x_0 - \varepsilon} \eta_0'(x) = \lim_{x \searrow x_0 - \varepsilon} \eta_0'(x) = 0,$$

and, similarly,

$$\lim_{x \nearrow x_0 + \varepsilon} \eta_0'(x) = \lim_{x \searrow x_0 + \varepsilon} \eta_0'(x) = 0.$$

These calculations prove that the function $\eta_0(x) \in C^1[a, b]$. Taking into account the above initial conditions satisfied by the function $\eta_0(x)$ we conclude that $\eta_0(x)$ satisfies all conditions from the hypotheses of our theorem. Therefore, we must have

$$\int_a^b \frac{\partial^2 L}{\partial y'^2} \eta'^2(x) \mathrm{d}x \geq 0,$$

the integral being computed for the function $y(x) + \varepsilon\eta_0(x)$ belonging to the considered vicinity of the function $y(x)$. We will prove that it is a contradiction.

On the other hand, we have

$$\frac{\partial^2 L}{\partial y^2} - \frac{d}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y \partial y'}\right) \in C^0.$$

Also, this function is defined on the closed (thus, bounded) interval $[a, b]$, such that by using the classical Weierstrass's theorem we deduce that it is bounded on the interval $[a, b]$ and there exists M, defined as follows

$$M=\sup_{a\le x\le b}\left|\frac{\partial^2 L}{\partial y^2}-\frac{d}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y\partial y'}\right)\right|.$$

Now, we can contradict the hypothesis

$$\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0}\ge 0.$$

Indeed, using the above considerations, we obtain

$$\left|\left.\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}\right|_{\varepsilon=0}\right|=\left|\int_a^b\left\{\left[\frac{\partial^2 L}{\partial y^2}-\frac{d}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y\partial y'}\right)\right]\eta^2(x)+\frac{\partial^2 L}{\partial y'^2}\eta'^2(x)\right\}\mathrm{d}x\right|\le$$

$$\le M\int_a^b\sin^4\frac{x^2-\varepsilon^2}{\varepsilon^2}\mathrm{d}x-\frac{\alpha^2}{2}\int_a^b 4\sin^2\frac{x^2-\varepsilon^2}{\varepsilon^2}\cos^2\frac{x^2-\varepsilon^2}{\varepsilon^2}\frac{4x^2}{\varepsilon^2}\mathrm{d}x\le$$

$$\le\int_{x_0-\varepsilon}^{x_0+\varepsilon}\mathrm{d}x-\frac{8\alpha^2}{\varepsilon^2}\int_{x_0-\varepsilon}^{x_0+\varepsilon}x^2\mathrm{d}x=2M\varepsilon-64\alpha^2\frac{x_0^2}{\varepsilon}.$$

Thus, considering, as usual, ε sufficient small, the second variation of the functional,

$$\frac{\delta^2 I(\varepsilon)}{\delta\varepsilon^2}$$

becomes negative that contradicts the hypothesis.

The theorem is proved. ■

Application. Using the Legendre's result, let us find the effective point of minimum for the following functional

$$I(y)=\int_0^{\pi/2}\left(y'^2-y^2\right)\mathrm{d}x,\ \ y(0)=y(\frac{\pi}{2})=1.$$

Using the Euler's equation, we obtain the following differential equation

$$y''+y=0.$$

Because the characteristic equation $r^2+1=0$ has the complex conjugated roots $\pm i$ we deduce that the solution of the differential equation is

$$y(x)=A\cos x+B\sin x.$$

Using the initial condition $y(0) = y(\frac{\pi}{2}) = 1$, it follows $A = 1$ and $B = 1$ such that

$$y(x) = \cos x + \sin x.$$

We will write the Lagrangean of our functional as a function of $y'(x)$. By direct calculations

$$y'(x) = \cos x - \sin x \Rightarrow y'^2(x) = 1 - \sin 2x.$$

The Lagrangean becomes

$$L\left(x, y(x), y'(x)\right) = y'^2(x) - y^2(x) = -2\sin 2x.$$

But $\sin 2x = 1 - 2y'(x)$ such that we have

$$L\left(x, y, y'\right) = 2y'^2 - 2 \Rightarrow \frac{\partial L}{\partial y'} = 4y' \Rightarrow \frac{\partial^2 L}{\partial y'^2} = 4 > 0.$$

According to the Legendre's result we are led to the conclusion that the function

$$y(x) = \cos x + \sin x$$

is an effective point of minimum for the given functional.

5.5 Isoperimetric Problems

There exists certain problems of variational calculus for which the function, that must extremate a functional, is subjected to some restrictions. We will call such a problem as a problem of conditional extremum, or, an isoperimetric problem. From among all kind of restrictions that can be imposed, we use the following one: Define a new functional, say $J(y(x))$, with other Lagrangean, say $M(x, y(x), y'(x))$, and consider only the functions $y(x)$ along which the new functional assumes a given value l.

Thus, together with the functional

$$I(y) = \int_a^b L\left(x, y(x), y'(x)\right) \mathrm{d}x$$

we consider the new functional

$$J(y) = \int_a^b M\left(x, y(x), y'(x)\right) \mathrm{d}x$$

and an isoperimetric problem can be formulated as follows:

From among all curves $y = y(x) \in C^1[a, b]$ along which the functional $J(y)$ assumes a given value l, determine the one for which the functional $I(y)$ assumes an extremal value.

With regard to the Lagrangeans L amd M we assume that they have continuous first and second partial derivatives for $a \le x \le b$ and for arbitrary values of $y(x)$ and $y'(x)$.

A well known isoperimetric problem is the *Dido's problem*, called also, the *Fisher's problem*. Among closed curves of length l, find the one that bounds the largest area. In this case, the Lagrangeans L and M are

$$L\left(x, y(x), y'(x)\right) = y(x), \;\; M\left(x, y(x), y'(x)\right) = \sqrt{1 + y'^2(x)}.$$

Consequently, we must find the curve $y = y(x)$ along which the functional

$$J(y) = \int_a^b \sqrt{1 + y'^2(x)}\mathrm{d}x$$

assumes a given value l (the length of the trawl!) and for which the functional

$$I(y) = \int_a^b y(x)\mathrm{d}x$$

assumes an extremal value.

We turn to the general isoperimetric problem and prove the main result in this context, due to Euler.

Theorem 5.5.1 *If a curve $y = y(x)$ extremizes the functional*

$$I(y) = \int_a^b L\left(x, y(x), y'(x)\right) \mathrm{d}x$$

subject to the conditions

$$J(y) = \int_a^b M\left(x, y(x), y'(x)\right) \mathrm{d}x = l, \;\; y(a) = y_a, \;\; y(b) = y_b$$

and $y = y(x)$ is not an extremal of the functional J, then there exists a constant λ such that the curve $y = y(x)$ is an extremal of the functional

$$\tilde{I}(y) = \int_a^b \left[L\left(x, y(x), y'(x)\right) - \lambda M\left(x, y(x), y'(x)\right)\right] \mathrm{d}x.$$

Proof Together with the function $y(x)$ consider a vicinity of functions of the form

$$\{y(x) + \alpha\eta(x) + \beta\gamma(x)\}_{\alpha,\beta}.$$

Every function from this vicinity has the same ends as $y(x)$, that is,

$$\eta(a) = \eta(b) = 0, \;\; \gamma(a) = \gamma(b) = 0.$$

If we compute the value of the functional I among one arbitrary representative of this vicinity, we find a function which depends of α and β:

$$I(y(x) + \alpha\eta(x) + \beta\gamma(x)) =$$

$$= \int_a^b L\left(x, y(x) + \alpha\eta(x) + \beta\gamma(x), y'(x) + \alpha\eta'(x) + \beta\gamma'(x)\right) \mathrm{d}x = I(\alpha, \beta).$$

But α and β are not independent, because

$$J(y(x) + \alpha\eta(x) + \beta\gamma(x)) =$$

$$= \int_a^b M\left(x, y(x) + \alpha\eta(x) + \beta\gamma(x), y'(x) + \alpha\eta'(x) + \beta\gamma'(x)\right) \mathrm{d}x = J(\alpha, \beta).$$

Thus,

$$J(\alpha, \beta) = l.$$

If we assume that J depends on β, we can use the theorem of the implicite functions such that we obtain the following three statements:

(1) β can be expressed as a function of α, i.e. $\beta = \beta(\alpha)$;
(2) if $\alpha = 0$ then $\beta = 0$, that is $\beta(0) = 0$;
(3) we can compute the derivative of β as follows

$$\beta'(\alpha) = \frac{d\beta}{d\alpha} = -\frac{\partial J/\partial\alpha}{\partial J/\partial\beta}.$$

For $\alpha = 0$, as such, $\beta = 0$, the representative of the vicinity reduces to the curve $y(x)$ that extremizes the functional I. That means that $\alpha = 0$ is the extremal of the function $I(\alpha, \beta) = I(\alpha, \beta(\alpha))$, and, according to the classical condition of the extremum, we have

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0. \Rightarrow$$

$$\Rightarrow \left(\frac{\partial I}{\partial \alpha} + \frac{\partial I}{\partial \beta}\frac{d\beta}{d\alpha}\right)_{\alpha=0} = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \left[\frac{\partial L}{\partial(y + \alpha\eta + \beta\gamma)}\eta(x) + \frac{\partial L}{\partial(y' + \alpha\eta' + \beta\gamma')}\eta'(x)\right] \mathrm{d}x\Bigg|_{\alpha=0} +$$

$$+ \int_a^b \left[\frac{\partial L}{\partial(y + \alpha\eta + \beta\gamma)}\gamma(x) + \frac{\partial L}{\partial(y' + \alpha\gamma' + \beta\gamma')}\gamma'(x)\right] \mathrm{d}x\frac{d\beta}{d\alpha}\Bigg|_{\alpha=0} = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \left[\frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x)\right] \mathrm{d}x + \int_a^b \left[\frac{\partial L}{\partial y}\gamma(x) + \frac{\partial L}{\partial y'}\gamma'(x)\right] \mathrm{d}x\frac{d\beta}{d\alpha} = 0$$

$$\Rightarrow \int_a^b \frac{\partial L}{\partial y}\eta(x)\mathrm{d}x + \int_a^b \frac{\partial L}{\partial y'}\eta'(x)\mathrm{d}x +$$

$$+ \left(\int_a^b \frac{\partial L}{\partial y}\gamma(x)\mathrm{d}x \int_a^b \frac{\partial L}{\partial y'}\gamma'(x)\mathrm{d}x\right)\frac{d\beta}{d\alpha} = 0 \tag{5.5.1}$$

Integrating by parts, we obtain

$$\int_a^b \frac{\partial L}{\partial y'}\eta'(x)\mathrm{d}x = \left.\frac{\partial L}{\partial y'}\eta\right|_a^b - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x,$$

and, since $\eta(a) = \eta(b) = 0$, it results

$$\int_a^b \frac{\partial L}{\partial y'}\eta'(x)\mathrm{d}x = -\int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x.$$

Similarly,

$$\int_a^b \frac{\partial L}{\partial y'}\gamma'(x)\mathrm{d}x = \frac{\partial L}{\partial y'}\gamma\bigg|_a^b - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\gamma(x)\mathrm{d}x,$$

and, since $\gamma(a) = \gamma(b) = 0$, it results

$$\int_a^b \frac{\partial L}{\partial y'}\gamma'(x)\mathrm{d}x = -\int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\gamma(x)\mathrm{d}x.$$

Taking into account these estimations, the conditions of extremum (5.5.1) become

$$\int_a^b \frac{\partial L}{\partial y}\eta(x)\mathrm{d}x - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x + \left(\int_a^b \frac{\partial L}{\partial y}\gamma(x)\mathrm{d}x - \int_a^b \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\gamma(x)\mathrm{d}x\right)\frac{d\beta}{d\alpha} = 0 \Rightarrow$$

$$\Rightarrow \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\mathrm{d}x + \left(\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\gamma(x)\right]\mathrm{d}x\right)\frac{d\beta}{d\alpha} = 0.$$

But,

$$\frac{d\beta}{d\alpha} = -\frac{\partial J/\partial\alpha}{\partial J/\partial\beta},$$

and then the previous relation becomes

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\mathrm{d}x - \left(\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\gamma(x)\right]\mathrm{d}x\right)\frac{\partial J/\partial\alpha}{\partial J/\partial\beta} = 0. \tag{5.5.2}$$

On the other hand, integrating by parts and taking into account that $\eta(a) = \eta(b) = 0$, we obtain

$$\frac{\partial J}{\partial\alpha} = \int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial M}{\partial y'}\right)\right]\eta(x)\mathrm{d}x.$$

In the same manner, taking into account that $\gamma(a) = \gamma(b) = 0$, we obtain

$$\frac{\partial J}{\partial\beta} = \int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial M}{\partial y'}\right)\right]\gamma(x)\mathrm{d}x.$$

Taking into account these estimations in Eq. (5.5.2), we obtain

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\mathrm{d}x -$$

$$-\int_a^b \frac{\int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial M}{\partial y'}\right)\right]\eta(x)\mathrm{d}x}{\int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial M}{\partial y'}\right)\right]\gamma(x)\mathrm{d}x}\left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\gamma(x)\mathrm{d}x = 0.$$

If we use the notation

$$\lambda = \frac{\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\gamma(x)\mathrm{d}x}{\int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial M}{\partial y'}\right)\right]\gamma(x)\mathrm{d}x}$$

the previous relation can be restated as follows

$$\int_a^b \left\{\left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right] - \lambda\left[\frac{\partial M}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial M}{\partial y'}\right)\right]\right\}\eta(x)\mathrm{d}x = 0.$$

Taking into account that $\eta(x)$ satisfies the conditions from the fundamental lemma, we obtain the following equation

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) - \lambda\left[\frac{\partial M}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial M}{\partial y'}\right)\right] = 0,$$

which can be restated in the following form

$$\frac{\partial}{\partial y}(L - \lambda M) - \frac{d}{\mathrm{d}x}\left[\frac{\partial}{\partial y'}(L - \lambda M)\right] = 0.$$

Finally, we observe that this equation is the Euler's equation for the functional $\tilde{I}(y)$, where

$$\tilde{I}(y) = \int_a^b (L - \lambda M)\,\mathrm{d}x,$$

and the theorem is concluded. ■

Remark. The parameter λ is called the *Lagrange's multiplier* and it is unknown. We can determine its value from the equation

$$\int_a^b M\left(x, y(x), y'(x)\right) \mathrm{d}x = l,$$

after we introduce in the Lagrangean the founded expression of the function $y(x)$.

Application 1. Let us solve the Dido's problem.

First of all, note that the curve must be convex. Indeed, if that were not so, there would be a straight line L such that if a portion of the boundary is reflected in the line, then we obtain a region of greater area than the original region having the same length of the boundary line.

Further note that any straight line that bisects a closed curve bounding a maximum area will also divide the area in half. Suppose the opposite line L_1 does not have this property. Then, by making a mirror reflection about L_1 of that portion of the surface with the greater area, we would obtain a curve of the same length but it would bound a greater area.

Choosing for the x-axis any of the straight lines that bisects the curve, we arrive at the following statement of the problem.

Find a curve $y = y(x)$, $y(-a) = y(a) = 0$, which together with the segment $-a \le x \le a$, for a given length $l > 2a$, bounds a maximum area. Thus, the problem has reduced to seeking the extremum of the functional

$$I(y) = \int_{-a}^{a} y(x), \ \ y(-a) = y(a) = 0$$

subjected to the accessory condition that

$$J(y) = \int_{-a}^{a} \sqrt{1 + y'^2(x)} \mathrm{d}x = l, \ \ l > 2a.$$

We form the auxiliary Lagrangean

$$\tilde{L}\left(x, y(x), y'\right) = L\left(x, y(x), y'\right) + \lambda M\left(x, y(x), y'\right) = y(x) + \lambda\sqrt{1 + y'^2(x)}$$

and consider the auxiliary functional

$$\tilde{I}(y) = \int_{-a}^{a} \tilde{L}\left(x, y(x), y'\right) \mathrm{d}x.$$

The Euler's equation for this functional is

$$\frac{d}{\mathrm{d}x}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1,$$

whence

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + C, \ \ C = \text{constant}.$$

Solving this equation for y', we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + C_1}{\sqrt{\lambda^2 - (x + C_1)^2}}.$$

Integrating this equation, we obtain

$$(x + C_1) + (y + C_2) = \lambda^2,$$

that is, a circle of radius λ with center at the point $(-C_1, -C_2)$. We determine the constants C_1, C_2 and the parameter λ from the boundary conditions $y(-a) = y(a) = 0$ and the isoperimetric condition, that is, equating the value of the functional $J(y)$ by l. We obtain

$$C_2^2 = \lambda^2 - (C_1 - a)^2$$

$$C_2^2 = \lambda^2 - (C_1 + a)^2$$

whence

$$C_1 = 0, \ \ C_2 = \sqrt{\lambda^2 - a^2},$$

such that

$$y(x) = \sqrt{\lambda^2 - x^2} - \sqrt{\lambda^2 - a^2},$$

$$y'(x) = -\frac{x}{\sqrt{\lambda^2 - x^2}}.$$

The isoperimeric condition yields

$$l = \int_{-a}^{a} \frac{\lambda}{\sqrt{\lambda^2 - x^2}} = \lambda \arcsin \frac{x}{\lambda}\Big|_{-a}^{a} = 2\lambda \arcsin \frac{a}{\lambda},$$

whence

$$\frac{a}{\lambda} = \sin \frac{l}{2\lambda}.$$

Solving this equation for λ, we find a certain value $\lambda = \lambda_0$ and then also the quantity

$$C_2 = \sqrt{\lambda_0^2 - a^2}.$$

Let us prove that the last equation always has a solution. Indeed, setting $l/2\lambda = t$, we reduce this equation to the form

$$\sin t = \frac{2a}{l}t,$$

where $2a/l = \alpha < 1$ by the statement of the problem. At the point $t = 0$, the function $y(t) = \sin t$ has the tangent slope $\pi/4$, while the function $y(t) = \alpha t$ has a smaller slope. Hence, the graphs of these functions have at least one point of intersection other than $O(0, 0)$.

Application 2. Let us solve the *geodesics's problem*.

Vector Solution. First of all, we remember that a geodesic of a surface is a curve of shortest length lying on a given surface and joining two given points of the surface. In our case, from among all curves on a sphere of radius 1 that joins two given points, find the shortest curve.

Suppose the sphere is given by the vector equation

$$\mathbf{r} = \mathbf{r}(u, v).$$

The equations of geodesics may be obtained as Euler's equations corresponding to the variational problem of finding the shortest distance on a surface between two given points. Let φ be the longitude, θ the latitude of a point on the sphere, and $\varphi = \varphi(\theta)$ be the equation of the desired curve. We then have

$$\mathbf{r} = \mathbf{r}(\varphi, \theta) = x(\varphi, \theta)\mathbf{i} + y(\varphi, \theta)\mathbf{j} + z(\varphi, \theta)\mathbf{k}.$$

Let us remember that the element of arc on a surface has the form

$$ds = \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\mathrm{d}\theta$$

and then the length of the curve between the points corresponding to the values θ_1 and θ_2 of the parameter θ is

$$J(u, v) = \int_{\theta_1}^{\theta_2} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\mathrm{d}\theta,$$

where E, F, G are coefficients of the quadratic form of the surface, that is,

$$E = \left(\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial u}\right), \ F = \left(\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right), \ G = \left(\frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial v}\right).$$

Here, $(\mathbf{a}, \mathbf{b})$ is the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

In the case of the sphere,

$$E = \left(\mathbf{r}_\varphi, \mathbf{r}_\varphi\right) = \sin^2\theta, \ G = (\mathbf{r}_\theta, \mathbf{r}_\theta) = 1, \ F = \left(\mathbf{r}_\varphi, \mathbf{r}_\theta\right) = 0,$$

and

$$J(\varphi, \theta) = \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta . \varphi'^2(\theta)} \mathrm{d}\theta.$$

The Lagrangean does not contain the function $\varphi(\theta)$ such that the Euler's equation reduces to its particular prime integral

$$\frac{d}{\mathrm{d}\theta} L_{\varphi'} = 0,$$

where

$$L_{\varphi'} = \frac{\sin^2\theta\varphi'(\theta)}{\sqrt{1 + \sin^2\theta . \varphi'^2(\theta)}},$$

such that

$$\frac{\sin^2\theta\varphi'(\theta)}{\sqrt{1 + \sin^2\theta . \varphi'^2(\theta)}} = C_1.$$

From this we get

$$\varphi'(\theta) = \frac{C_1}{\sin\theta\sqrt{\sin^2\theta - C_1^2}} =$$

$$= \frac{C_1}{\sin^2\theta\sqrt{(1 - C_1^2) - C_1^2\cot^2\theta}} = -\frac{C_1 d(\cot^2\theta)}{\sqrt{(1 - C_1^2) - C_1^2\cot^2\theta}}.$$

Integrating we obtain

$$\varphi(\theta) = \arccos\frac{C_1\cot^2\theta}{\sqrt{1 - C_1^2}} + C_2,$$

that is,

$$\varphi(\theta) = \arccos(C\cot\theta) + C_2, \ \text{where} \ C = \frac{C_1}{\sqrt{1 - C_1^2}},$$

whence

$$C \cot \theta = \cos[\varphi(\theta) - C_2],$$

or, equivalently,

$$\cot \theta = A \cos \varphi(\theta) + B \sin \varphi(\theta)$$

where

$$A = \frac{\cos C_2}{C}, \;\; B = \frac{\sin C_2}{C}.$$

Multiplying both members by $\sin \theta$, we obtain

$$\cos \theta = A \cos \varphi \sin \theta + B \sin \varphi \sin \theta,$$

from where, passing to Cartesian coordinates, we deduce

$$z = Ax + By.$$

This is the equation of the plane passing through the center of the sphere and intersecting the sphere along a great circle. Thus, the geodesic of the sphere is a great circle.

Cartesian Solution. Let us consider the sphere having the origin as center and the radius equal to 1. From along all curves that join two arbitrary points A and B on the sphere, find that of shortest length. Consider the curve in the parametric form

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t), \;\; t \in [a, b]. \end{cases}$$

The ends a and b of the interval will be determined by the fact that the arc of the curve passes through points A and B. As we know, the element of the arc is

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\mathrm{d}t.$$

The curve between A and B has the length

$$s(x, y, t) = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\mathrm{d}t.$$

So, we must extremise this functional taking into account that the arc of the curve must lie on the sphere $x^2 + y^2 + z^2 = 1$. Based on Theorem 5.5.1, we have the functional

$$I(x, y, z) = \int_a^b \left[\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \lambda\left(x^2 + y^2 + z^2 - 1\right)\right] \mathrm{d}t,$$

that leads to the system of equations

$$\frac{\partial L}{\partial x} - \frac{d}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

$$\frac{\partial L}{\partial z} - \frac{d}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0,$$

where the Lagrangean L is

$$L\left(t, x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)\right) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \lambda\left(x^2 + y^2 + z^2 - 1\right).$$

Since

$$\frac{\partial L}{\partial x} = -2\lambda x,\ \frac{\partial L}{\partial y} = -2\lambda y,\ \frac{\partial L}{\partial z} = -2\lambda z$$

and

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}},\ \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}},\ \frac{\partial L}{\partial \dot{z}} = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}},$$

the above system of equation becomes

$$-2\lambda x - \frac{d}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = 0$$

$$-2\lambda y - \frac{d}{\mathrm{d}t}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = 0$$

$$-2\lambda z - \frac{d}{\mathrm{d}t}\left(\frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = 0,$$

which can be restated as follows

$$\frac{d}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + 2\lambda x = 0$$

$$\frac{d}{\mathrm{d}t}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)+2\lambda y=0$$

$$\frac{d}{\mathrm{d}t}\left(\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)+2\lambda z=0.$$

It is strong to solve this system in the variable t, such that we pass to the new variable s, taking into account that

$$ds=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\mathrm{d}t.$$

By direct calculations, we obtain

$$\dot{x}=\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{ds}\frac{ds}{\mathrm{d}t}=x'\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\Rightarrow\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}=x',$$

$$\dot{y}=\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{ds}\frac{ds}{\mathrm{d}t}=y'\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\Rightarrow\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}=y',$$

$$\dot{z}=\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\mathrm{d}z}{ds}\frac{ds}{\mathrm{d}t}=z'\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\Rightarrow\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}=z'.$$

Then the above system becomes

$$\frac{d}{\mathrm{d}t}\left(x'\right)+2\lambda x=0,$$

$$\frac{d}{\mathrm{d}t}\left(y'\right)+2\lambda y=0,$$

$$\frac{d}{\mathrm{d}t}\left(z'\right)+2\lambda z=0,$$

that is,

$$\frac{\mathrm{d}x'}{ds}\frac{ds}{\mathrm{d}t}+2\lambda x=0,$$

$$\frac{\mathrm{d}y'}{ds}\frac{ds}{\mathrm{d}t}+2\lambda y=0,$$

$$\frac{\mathrm{d}z'}{ds}\frac{ds}{\mathrm{d}t}+2\lambda z=0,$$

from where we deduce

$$x'' = -\frac{2\lambda}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} x,$$

$$y'' = -\frac{2\lambda}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} y,$$

$$z'' = -\frac{2\lambda}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} z.$$

Using the notation

$$\frac{2\lambda}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \beta(t),$$

the last system becomes

$$\begin{cases} x'' = -\beta(t)x, \\ y'' = -\beta(t)y, \\ z'' = -\beta(t)z. \end{cases} \tag{5.5.3}$$

On the other hand, by derivation the equation of the sphere, with respect to s, $x^2 + y^2 + z^2 - 1 = 0$, we obtain

$$xx' + yy' + zz' = 0,$$

and, after one more time derivation

$$x'^2 + y'^2 + z'^2 + xx'' + yy'' + zz'' = 0. \tag{5.5.4}$$

Multiplying the first equation from (5.5.3) by x, the second by y and the third by z and adding the resulting relations, we obtain

$$xx'' + yy'' + zz'' = -\beta(t)\left(x^2 + y^2 + z^2\right) = -\beta(t).$$

Introducing this result in Eq. (5.5.4) it follows

$$x'^2 + y'^2 + z'^2 = \beta(t). \tag{5.5.5}$$

On the other hand, using the above proved system of relations

$$\begin{cases} \dot{x} = x'\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \\ \dot{y} = y'\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \\ \dot{z} = z'\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \end{cases}$$

from where it is easy to see that

$$x'^2 + y'^2 + z'^2 = 1$$

and then from Eq. (5.5.5) we deduce

$$\beta(t) = 1,$$

and the system (5.5.3) becomes

$$\begin{cases} x'' + x = 0, \\ y'' + y = 0, \\ z'' + z = 0. \end{cases}$$

It is easy to solve the system of differential equations (with the aid of the characteristic equation) and find the solutions

$$\begin{cases} x(s) = A_1 \cos s + A_2 \sin s, \\ y(s) = B_1 \cos s + B_2 \sin s, \\ z(s) = C_1 \cos s + C_2 \sin s, \end{cases}$$

where A_1, A_2, B_1, B_2, C_1, C_2 are constants.

From the first two equations we find the expression for $\cos s$ and $\sin s$ and introducing in the last equation, we obtain an equation of the form

$$z = Ax + By, \quad A, \quad B = \text{constants},$$

that is, the equation of the plane passing through the center of the sphere and intersecting the sphere along a great circle. Thus, the geodesic of the sphere is a great circle.

5.6 Moving Boundary Problems

In all problems of variational calculus that we already studied, the endpoints of the function that extremizes a functional there were fixed. In this paragraph we introduce some notions with regard to moving boundary problems. We will consider only two cases:

(1) An endpoint of the extremal (for instance, left-hand end of the curve) is fixed and the other (the right-hand end) is movable along a given curve $y = \varphi(x)$.
(2) The left-hand end of the extremal is movable along a given curve $y = \varphi(x)$ and the right-hand end is movable along a given curve $y = \psi(x)$.

Of course, there exists more complicated problems when the endpoints of the extremal are movable along, either a given curve or a certain surface, or, more general, both endpoints are movable along two given surfaces.

For the moment, we will consider the problem of finding the shortest distance between a fixed point $A(a, b)$ and a curve

$$(C) \qquad\qquad y = \varphi(x), \quad x \in [\alpha, \beta].$$

As such, an endpoint of the extremal will be fixed and the other is movable along the curve (C). Denote by $B(x_1, \varphi(x_1)$ the point where it attains the minimum of distance between point A and the curve (C). In this case the functional that must be extremized is

$$I(y) = \int_a^{x_1} L\left(x, y(x), y'(x)\right) \mathrm{d}x. \tag{5.6.1}$$

Theorem 5.6.1 *If the function $y(x)$ extremizes the functional $I(y)$ from Eq. (5.2.2), then it satisfies the following system of equations*

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0 \tag{5.6.2}$$

$$L + \frac{\partial L}{\partial y'}\left(\frac{d\varphi}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=x_1} = 0. \tag{5.6.3}$$

Proof First of all, we must outline that the relation (5.6.1) is called the *transversality condition*.

Consider a vicinity of first order of function $y(x)$

$$\{y(x) + \varepsilon\eta(x)\}_\varepsilon, \ \varepsilon > 0$$

and compute the value of the functional $I(y)$ for an arbitrary representative of this vicinity

$$I(y + \varepsilon\eta) = \int_a^{x_1} L\left(x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x)\right) \mathrm{d}x = I(\varepsilon).$$

For an arbitrary representative of this vicinity (that is, for an arbitrary fixed ε), an equality of the form

$$y(x) + \varepsilon\eta(x) = \varphi(x), \tag{5.6.4}$$

defines the value of the point x_1 as an intersection of those curves. Of course, for arbitrary ε, we obtain $x_1 = x_1(\varepsilon)$ and the functional can be written in the form

$$I(y+\varepsilon\eta) = \int_a^{x_1(\varepsilon)} L\left(x, y(x)+\varepsilon\eta(x), y'(x)+\varepsilon\eta'(x)\right)\mathrm{d}x = I(\varepsilon).$$

It is obvious that for $\varepsilon = 0$ the representative of the vicinity reduces to the function $y(x)$ that extremizes the functional. Consequently, $\varepsilon = 0$ is the point of extremum for the function $I(\varepsilon)$ and, therefore,

$$\left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

Using the rule of derivation for an integral having parameter, it follows

$$\left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\mathrm{d}x_1(\varepsilon)}{d\varepsilon} L\left(x_1(\varepsilon), y(x_1(\varepsilon)), y'x_1(\varepsilon)\right)\right|_{\varepsilon=0} +$$

$$+ \left.\int_a^{x_1(\varepsilon)} \left(\frac{\partial L}{\partial(y+\varepsilon\eta)}\eta(x) + \frac{\partial L}{\partial(y'+\varepsilon\eta')}\eta'(x)\right)\mathrm{d}x\right|_{\varepsilon=0} = 0.$$

Taking into account that $\varepsilon = 0$, we can write

$$\frac{\mathrm{d}x_1(0)}{d\varepsilon} L\left(x_1(0), y(x_1(0)), y'x_1(0)\right) + \int_a^{x_1(0)} \left(\frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x)\right)\mathrm{d}x = 0. \tag{5.6.5}$$

Let us compute, by parts, the last integral in this equation

$$\int_a^{x_1(0)} \frac{\partial L}{\partial y'}\eta'(x)\mathrm{d}x = \left.\frac{\partial L}{\partial y'}\eta(x)\right|_a^{x_1(0)} - \int_a^{x_1(0)} \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x.$$

Taking into account that $\eta(a) = 0$, we have

$$\int_a^{x_1(0)} \frac{\partial L}{\partial y'}\eta'(x)\mathrm{d}x = \frac{\partial L}{\partial y'}\eta(x_1(0)) - \int_a^{x_1(0)} \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\mathrm{d}x.$$

Thus, the equation (5.5.5) becomes

$$\frac{\mathrm{d}x_1(0)}{d\varepsilon} L\left(x_1(0), y(x_1(0)), y'x_1(0)\right) + \frac{\partial L}{\partial y'}\eta(x_1(0)) +$$

$$+\int_{a}^{x_1(0)}\left[\frac{\partial L}{\partial y}-\frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\mathrm{d}x=0. \tag{5.6.6}$$

Now, we write the fact that $x_1(\varepsilon)$ verifies the equation (5.6.4), that is

$$y(x_1(\varepsilon))+\varepsilon\eta(x_1(\varepsilon))=\varphi(x_1(\varepsilon)). \tag{5.6.7}$$

Using the derivative with respect to ε in Eq. (5.6.7), it follows

$$\frac{\mathrm{d}y}{\mathrm{d}x_1}\frac{\mathrm{d}x_1}{d\varepsilon}+\eta(x_1(\varepsilon))+\varepsilon\frac{d\eta}{\mathrm{d}x_1}\frac{\mathrm{d}x_1}{d\varepsilon}=\frac{d\varphi}{\mathrm{d}x_1}\frac{\mathrm{d}x_1}{d\varepsilon}\Rightarrow$$

$$\Rightarrow\eta(x_1(\varepsilon))=\frac{\mathrm{d}x_1}{d\varepsilon}\left[\frac{d\varphi}{\mathrm{d}x_1}-\frac{\mathrm{d}y}{\mathrm{d}x_1}\right]-\varepsilon\frac{d\eta}{\mathrm{d}x_1}\frac{\mathrm{d}x_1}{d\varepsilon}$$

such that, putting $\varepsilon=0$, we obtain

$$\eta(x_1(0))=\frac{\mathrm{d}x_1(\varepsilon)}{d\varepsilon}\left(\frac{d\varphi}{\mathrm{d}x_1}-\frac{\mathrm{d}y}{\mathrm{d}x_1}\right)_{\varepsilon=0}. \tag{5.6.8}$$

Substituting this result in Eq. (5.6.6), we deduce

$$\frac{\mathrm{d}x_1(0)}{d\varepsilon}\left[L\left(x_1,y(x_1),y'x_1\right)+\frac{\partial L}{\partial y'}\left(\frac{d\varphi}{\mathrm{d}x}-\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=x_1}\right]+$$

$$+\int_{a}^{x_1}\left[\frac{\partial L}{\partial y}-\frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\mathrm{d}x=0. \tag{5.6.9}$$

The equality (5.6.9) holds for any η, because we have computed the value of the functional I for an arbitrary representative of the vicinity of function $y(x)$. So, we can choose that η for which $\eta(x_1(0))=0$ and the relation (5.6.8) becomes

$$\frac{\mathrm{d}x_1(\varepsilon)}{d\varepsilon}\left(\frac{d\varphi}{\mathrm{d}x_1}-\frac{\mathrm{d}y}{\mathrm{d}x_1}\right)_{\varepsilon=0}=0.$$

But, the last term cannot be zero, such that we deduce

$$\frac{\mathrm{d}x_1(0)}{d\varepsilon}=0,$$

and the equality (5.6.9) reduces to

$$\int_a^{x_1(0)} \left[\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\mathrm{d}x = 0.$$

Therefore, we can use the fundamental lemma and conclude

$$\frac{\partial L}{\partial y} - \frac{d}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0$$

that is, the Euler's equation (5.6.2).

Taking into account this conclusion, the relation (5.30) reduces to

$$L\left(x_1, y(x_1), y'x_1\right) + \frac{\partial L}{\partial y'}\left(\frac{d\varphi}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=x_1} = 0,$$

that is, the transversality condition from the enunciation of our theorem. So, both and the Euler's equation and the transversality condition are proved such that the theorem is concluded. ■

Let us give a geometrical interpretation of the transversality condition. To this, we prove the following proposition.

Proposition 5.6.1 *If the Lagrangean of the functional $I(y)$ has the form*

$$L = L\left(x, y(x), y'(x)\right) = h(x, y(x))\sqrt{1 + y'^2(x)},\ \ h(x, y) \neq 0,$$

and, as a consequence, the functional is

$$I(y) = \int_a^{x_1} h(x, y(x))\sqrt{1 + y'^2(x)}\mathrm{d}x,$$

then the transversality condition becomes

$$y'^2(x) = -\frac{1}{\varphi'^2(x)},$$

where $y(x)$ is the extremal of the functional and $\varphi(x)$ is the curve of one of the movable endpoints of the extremal.

Proof The general transversality condition

$$L\left(x_1, y(x_1), y'x_1\right) + \frac{\partial L}{\partial y'}\left(\frac{d\varphi}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=x_1} = 0,$$

becomes, in our case,

$$h(x, y(x))\sqrt{1 + y'^2(x)} + h(x, y(x))\frac{y'(x)}{\sqrt{1 + y'^2(x)}}\left(\varphi'(x) - y'(x)\right)_{x=x_1} = 0.$$

By hypothesis, $h(x, y) \neq 0$, such that we deduce

$$\frac{1 + y'^2(x) + y'(x)\varphi'(x) - y'^2(x)}{\sqrt{1 + y'^2(x)}} = 0 \Rightarrow$$

$$\Rightarrow 1 + y'(x)\varphi'(x) = 0,$$

such that desired result is proved. ■

Remark. The transversality condition from the above proposition asserts that the extremal of the functional and the curve where is movable one of the endpoints of the extremal, must be two orthogonal curves.

Application. Let us compute the minimum distance between the point $A(-1, 5)$ and the curve $y^2 = x$. In this case the functional is

$$I(y) = \int_{-1}^{x_1} \sqrt{1 + y'}\mathrm{d}x,$$

x_1 being the movable point on the curve

$$y = \varphi(x) = \sqrt{x}.$$

Instead of the Euler's equation we can use the prime integral

$$\frac{\partial L}{\partial y'} = C,\ \ C = \text{constant}.$$

The transversality condition received the form

$$\sqrt{1 + y'} + \frac{y'}{\sqrt{1 + y'^2}}\left(\frac{1}{2\sqrt{x}} - y'\right)_{x=x_1} = 0.$$

From

$$\frac{y'}{\sqrt{1 + y'^2}} = C \Rightarrow y' = C_1 \Rightarrow y = C_1x + C_2,$$

such that the transversality condition becomes

$$\sqrt{1 + C_1^2} + \frac{C_1}{\sqrt{1 + C_1^2}}\left(\frac{1}{2\sqrt{x_1}} - C_1\right) = 0 \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{1+C_1^2}} + \frac{1}{2\sqrt{x_1}} \frac{C_1}{\sqrt{1+C_1^2}} = 0 \Rightarrow 2\sqrt{x_1} + C_1 = 0.$$

Then, since the points $A(-1, 5)$ and $B(x_1, \sqrt{x_1})$ belonging to the straight line $y = C_1 x + C_2$ we obtain also two relations to determine $C_1 = -2$, $C_2 = 3$ and $x_1 = 1$. The straight line becomes $y = -2x + 3$ and then the minimum distance is

$$\int_{-1}^{1} \sqrt{1 + y'^2} \mathrm{d}x = \int_{-1}^{1} \sqrt{5} \mathrm{d}x = 2\sqrt{5}.$$

Chapter 6
Quasi-linear Equations

6.1 Canonical Form for $n = 2$

Let Ω be a bounded domain in the $n-$dimensional Euclidean space $\mathbb{R}^n$. The general form of a partial differential equation is:

$$F(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_i x_j}, u_{x_1 x_2 \ldots x_i}, \ldots, u_{x_1 x_2 \ldots x_n}) = 0, \quad (6.1.1)$$

where by $u_{x_i}, u_{x_i x_j}, u_{x_i x_j x_k}, \ldots,$ we have denoted partial derivatives

$$\frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \ldots$$

The unknown function is $u(x) = u(x_1, x_2, \ldots, x_n)$, $x \in \Omega$. The function F satisfies, with respect with its arguments, certain hypotheses which permit the mathematical operations that are necessary to solve the equation.

A real function $u(x)$ defined in the domain Ω, where Eq. (6.1.1) is considered, which is continuous together with its partial derivatives contained in the equation and which turns the equation into an identity is called a *regular solution* of the equation. We shall study the partial differential equations of the form (6.1.1) which contains only the partial derivatives until to the order two, inclusively.

Definition 6.1.1 A partial differential equation of order two, in a single unknown function, depending of n independent variables of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, x_2, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x_1, x_2, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) \quad (6.1.2)$$

is called a quasi-linear partial differential equation of order two.

The functions $a_{ij} = a_{ji}$ and f are known functions and are generally, assumed to be continuous with respect to its arguments.

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In all that following we consider only the case of two independent variables, ($n = 2$), such that the quasi-linear equation (6.1.2) becomes

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} = f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}). \tag{6.1.3}$$

The problem of the integration of an equation of the form (6.1.3) is the problem of the determination of a function $u = u(x, y)$, $u : \Omega \to \mathbb{R}$, $u \in C^2(\Omega)$ which verifies the given equation. In view of making the integration of Eq. (6.1.3) easier, we make a change of independent variables: from the variables (x, y) we pass to the variables (ξ, η), as follows

$$\begin{aligned} \xi &= \xi(x, y), \\ \eta &= \eta(x, y), \end{aligned} \tag{6.1.4}$$

such that $\xi, \eta \in C^2(\Omega)$ and

$$\left|\frac{\partial(\xi, \eta)}{\partial(x, y)}\right| = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0, \text{ on } \Omega. \tag{6.1.5}$$

The change (6.1.4) is a non-degenerate affine transformation of variables and is made with the hope that in the new partial differential equation, in the variables ξ and η, also of the form Eq. (6.1.3), one or two coefficients are zero.

Remark. Due to the hypotheses imposed to ξ and η, we can apply at any point from Ω the theorem of implicit functions. Then, if we arbitrarily fix $(x_0, y_0) \in \Omega$, we can find the solution of the system (6.1.4) with respect to the unknowns x and y, such that in a vicinity of the point (x_0, y_0), we obtain

$$\begin{aligned} x &= x(\xi, \eta) \\ y &= y(\xi, \eta). \end{aligned} \tag{6.1.6}$$

If we denote by $\xi_0 = \xi(x_0, y_0)$, $\eta_0 = \eta(x_0, y_0)$, then we have $x_0 = x(\xi_0, \eta_0)$ and $y_0 = y(\xi_0, \eta_0)$.

In order to obtain the partial differential equation in the new variables ξ and η, we substitute the derivatives of the unknown function by the derivatives with respect to the new variables. We have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

Then:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \\ + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \\ + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}, \tag{6.1.7}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \\ + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}.$$

If we substitute the derivatives from Eq. (6.1.7) in (6.1.3) we obtain

$$\overline{a_{11}}\frac{\partial^2 u}{\partial \xi^2} + 2\overline{a_{12}}\frac{\partial^2 u}{\partial \xi \partial \eta} + \overline{a_{22}}\frac{\partial^2 u}{\partial \eta^2} = F(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}), \tag{6.1.8}$$

where the new coefficients $\overline{a_{ij}}$ have the expressions:

$$\overline{a_{11}} = a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + a_{22}\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$\overline{a_{12}} = a_{11}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + a_{12}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + a_{22}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}, \tag{6.1.9}$$

$$\overline{a_{22}} = a_{11}\left(\frac{\partial \eta}{\partial x}\right)^2 + 2a_{12}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + a_{22}\left(\frac{\partial \eta}{\partial y}\right)^2.$$

It is clear that the annulment of the coefficients $\overline{a_{11}}$ and $\overline{a_{22}}$ of the Eq. (6.1.8) is connected to the solution of the partial differential equation of first order:

$$a_{11}\left(\frac{\partial z}{\partial x}\right)^2 + 2a_{12}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + a_{22}\left(\frac{\partial z}{\partial y}\right)^2 = 0. \tag{6.1.10}$$

Indeed, if $z = \varphi(x, y)$ is a solution of Eq. (6.1.10), then using the transformation of variables

$$\begin{aligned}\xi &= \varphi(x, y)\\ \eta &= \eta(x, y),\end{aligned}$$

where η is an arbitrary variable, but it must satisfy the condition (6.1.5), then, from Eq. (6.1.9) we deduce that $\overline{a_{11}} = 0$.

If we choose the new variables

$$\begin{aligned}\xi &= \xi(x, y)\\ \eta &= \varphi(x, y),\end{aligned}$$

where $z = \varphi(x, y)$ is a solution of Eq. (6.1.10), then from Eq. $(6.1.9)_3$ we obtain that $\overline{a_{22}} = 0$.

On the other hand, let us observe that the solution of the partial differential equation (6.1.10) is connected to the solution of the ordinary differential equation

$$a_{11}\,(\mathrm{d}y)^2 - 2a_{12}\mathrm{d}y\,\mathrm{d}x + a_{22}\,(\mathrm{d}x)^2 = 0, \tag{6.1.11}$$

which, formally, can be rewritten in the form:

$$a_{11}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2a_{12}\frac{\mathrm{d}y}{\mathrm{d}x} + a_{22} = 0. \tag{6.1.12}$$

Proposition 6.1.1 **(i)** *Let $\varphi(x, y) = C$ be a prime integral of the Eq. (6.1.11), where C is an arbitrary constant. Then the function $z = \varphi(x, y)$ is a solution of the Eq. (6.1.10).*
(ii) *Let $z = \varphi(x, y)$ be a solution of the Eq. (6.1.10). Then $\varphi(x, y) = C$, where C is an arbitrary constant, is a prime integral of the Eq. (6.1.11).*

Proof (i). Let $\varphi(x, y) = C$ be a prime integral of the Eq. (6.1.11).
Without restriction of the generality, we can assume that $\frac{\partial\varphi}{\partial y}(x, y) \neq 0$ on Ω.
If

$$\frac{\partial\varphi}{\partial y}(x, y) = 0, \forall(x, y) \in \Omega_0 \subset \Omega,$$

then we continue the study only on $\Omega\backslash\Omega_0$.

If

$$\frac{\partial\varphi}{\partial xy}(x, y) = 0, \forall(x, y) \in \Omega,$$

then we change the role of the variables x and y.

If both

$$\frac{\partial\varphi}{\partial x}(x, y) = 0, \text{ and } \frac{\partial\varphi}{\partial y}(x, y) = 0, \ \forall(x, y) \in \Omega,$$

then $\varphi(x, y)$ is a constant function and, therefore, the Eq. (6.1.10) has the null solution. Consequently, we can assume that $\frac{\partial\varphi}{\partial y} \neq 0$. Then in a vicinity of a point (x_0, y_0)

for which $\frac{\partial\varphi}{\partial y}(x_0, y_0) \neq 0$, we can write $y = f(x, c_0)$, where $c_0 = \varphi(x_0, y_0)$. Moreover,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial\varphi}{\partial x}(x, y)}{\frac{\partial\varphi}{\partial y}(x, y)}. \tag{6.1.13}$$

We substitute Eq. (6.1.13) in (6.1.12), which is equivalent to Eq. (6.1.11), and we obtain

$$0 = \left[a_{11}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2a_{12}\frac{\mathrm{d}y}{\mathrm{d}x} + a_{22}\right]_{(x_0, y_0)} =$$
$$= \left[a_{11}\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2 - 2a_{11}\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right) + a_{22}\right]_{(x_0, y_0)} =$$
$$= \left[a_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y} + a_{22}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\frac{1}{\left(\frac{\partial\varphi}{\partial y}\right)^2},$$

whence it follows

$$\left[a_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial x} + a_{22}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]_{(x_0, y_0)} = 0,$$

for all the possibilities to choosing (x_0, y_0) in Ω, that is, $\varphi(x, y)$ is a solution for the Eq. (6.1.10).

(ii). Let us assume that $z = \varphi(x, y)$ is a solution for the Eq. (6.1.10) and show that $\varphi(x, y) = C$, where C is an arbitrary constant, is a prime integral for the Eq. (6.1.11). To this we must show that $d\varphi(x, y) = 0$, that is,

$$\frac{\partial\varphi}{\partial x}\mathrm{d}x + \frac{\partial\varphi}{\partial y}\mathrm{d}y = 0,$$

whence it follows

$$\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}} = -\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{6.1.14}$$

If we write the fact that $z = \varphi(x, y)$ is a solution of the Eq. (6.1.10)

$$a_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial x} + a_{22}\left(\frac{\partial\varphi}{\partial y}\right)^2 = 0,$$

and we divide here, formally, by $\frac{\partial\varphi}{\partial y}$, without repeating the considerations regarding the theorem of implicit functions, which, obviously, is still valid, from the first part of the proof, we obtain

$$a_{11}\left(\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2+2a_{12}\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}+a_{22}=0.$$

Here, we substitute Eq. (6.1.14) such that the Eq. (6.1.12) is obtained. ■

Essentially, in the Proposition (6.1.1) it is shown that to find a solution for the Eq. (6.1.10) means to find the prime integrals of the Eq. (6.1.12).

Consequently, we have shown that to annul the coefficients $\overline{a_{11}}$ and $\overline{a_{22}}$ of the Eq. (6.1.8), means to find the prime integrals of the Eq. (6.1.12). The Eq. (6.1.11) is called *the equation of the characteristics* and its prime integrals are called characteristics or characteristic curves. Analyzing the equation of the characteristics (6.1.11), we ascertain that to find its prime integrals, we have three different cases, according to the discriminant Δ of the equation, $\Delta = a_{12}^2 - a_{11}a_{22}$:

1^o. If $\Delta > 0$, the Eq. (6.1.11) admits two real distinct characteristics curves. Then the partial differential equation is called the *hyperbolical equation*.

2^o. If $\Delta = 0$, the Eq. (6.1.11) admits only one real characteristic. Then the partial differential equation is called the *parabolical equation*.

3^o. If $\Delta < 0$, the Eq. (6.1.11) admits two complex conjugated characteristics. Then the partial differential equation is called the *elliptical equation*.

The above classification is made regarding the partial differential equation in its initial form (6.1.3). But, the transformation of coordinates (6.1.4) with the condition (6.1.5), does not affect the type of the Eq. (6.1.3).

Indeed, if we compute $\overline{\Delta}$ for the form (6.1.8) of the equation, (i.e. the canonical form), we ascertain that the discriminant Δ has the same sign:

$$\overline{\Delta}=\overline{a_{12}}^2-\overline{a_{11}}\,\overline{a_{22}}=\left(a_{12}^2-a_{11}a_{22}\right)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2,$$

that is,

$$\overline{\Delta}=\Delta\left(\frac{\partial(\xi,\eta)}{\partial(x,y)}\right)^2, \qquad (6.1.15)$$

where we take into account the form (6.1.9) of the coefficients $\overline{a_{ij}}$. From Eq. (6.1.15) we deduce that, if the equation is hyperbolical, parabolical or elliptical, in a system of coordinates, then it has the same type if we pass to another system of coordinates, if the transformation of coordinates is non-degenerate, that is, the condition (6.1.5) holds.

It is easy to ascertain that Δ is a continuous function with respect to the variables (x, y). It is well known fact that if a continuous function is positive in a point, then it is

positive in an entire vicinity of the respective point. So, we can divide the whole plane in three different sets. We shall call the domain of hyperbolicity for the Eq. (6.1.3) the set of the points from the plane $\mathbb{R}^2$ for which the Eq. (6.1.3) is hyperbolical. Analogically can be define the domains of parabolicity and of ellipticity, respectively. In the following we intend to find the canonical form of a partial differential equation in all three cases.

1^o **The hyperbolical case**: $\Delta = a_{12}^2 - a_{11}a_{22} > 0$. In this case the Eq. (6.1.11) has two real and distinct prime integrals: $\varphi(x, y) = C_1$, $\psi(x, y) = C_2$, where C_1 and C_2 are arbitrary constants. Consider the new variables (ξ, η) in the form

$$\xi = \varphi(x, y), s\eta = \psi(x, y)$$

and, based on the Proposition 6.1.1, we will obtain that $\overline{a_{11}} = 0$ and $\overline{a_{22}} = 0$, such that the canonical form of the hyperbolical equation is

$$\overline{a_{12}}\frac{\partial^2 u}{\partial\xi\partial\eta} = \overline{f}\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right)$$

or, if we divide by $\overline{a_{12}}$ (which obviously, can not be null):

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = F\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right). \tag{6.1.16}$$

Let us observe that the transformation (6.1.16) is non-degenerate. Indeed,

$$\begin{aligned}
&\left|\frac{\partial(\xi, \eta)}{\partial(x, y)}\right| = 0 \Leftrightarrow \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial y} = 0 \Leftrightarrow \\
&-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}} = -\frac{\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}} \Leftrightarrow \frac{a_{12} + \sqrt{\Delta}}{a_{11}} = \frac{a_{12} - \sqrt{\Delta}}{a_{11}} \Leftrightarrow \\
&\Leftrightarrow \sqrt{\Delta} = -\sqrt{\Delta} \Leftrightarrow \Delta = 0,
\end{aligned}$$

which is absurd, taking into account that $\Delta > 0$.

2^o **The parabolical case**: $\Delta = a_{12}^2 - a_{11}a_{22} = 0$. In this case the characteristic equation (6.1.11) has only one real prime integral, $\varphi(x, y) = C$, where C is an arbitrary constant. We can use the new variables (ξ, η) as follows

$$\xi = \varphi(x, y), \eta = \eta(x, y),$$

where η is an arbitrary function of the class C^2, which together with φ assures the condition that the transformation (6.1.18) is non-degenerate

$$\left|\frac{\partial(\xi, \eta)}{\partial(x, y)}\right| = \frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\eta}{\partial x}\frac{\partial\varphi}{\partial y} \neq 0. \tag{6.1.17}$$

Since we have chosen $\xi = \varphi(x, y)$, based on the Proposition 6.1.1, we deduce that $\overline{a_{11}} = 0$. Let us prove that and $\overline{a_{12}} = 0$.

Proposition 6.1.2 *If ξ and η are of the form (6.1.18), and satisfy the condition (6.1.19), then we have $\overline{a_{12}} = 0$.*

Proof From $a_{12}^2 = a_{11}a_{22}$ we deduce that a_{11} and a_{22} have, simultaneous, the same sign and, without loss of generality we assume that $a_{11} > 0$ and $a_{22} > 0$. Then $a_{12} = \pm\sqrt{a_{11}}\sqrt{a_{22}}$. According to the Proposition 6.1.1, it results that $\overline{a_{11}} = 0$ and, therefore

$$0 = a_{11}\left(\frac{\partial\xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + a_{11}\left(\frac{\partial\xi}{\partial y}\right)^2 =$$
$$= \left(\sqrt{a_{11}}\frac{\partial\xi}{\partial x}\right)^2 \pm \sqrt{a_{11}}\sqrt{a_{22}}\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + \left(\sqrt{a_{22}}\frac{\partial\xi}{\partial x}\right)^2 =$$
$$= \left(\sqrt{a_{11}}\frac{\partial\xi}{\partial x} \pm \sqrt{a_{22}}\frac{\partial\xi}{\partial y}\right)^2.$$

This implies that

$$\sqrt{a_{11}}\frac{\partial\xi}{\partial x} \pm \sqrt{a_{22}}\frac{\partial\xi}{\partial y} = 0. \tag{6.1.18}$$

Using Eq. (6.1.9) we deduce

$$\overline{a_{12}} = a_{11}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + a_{12}\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + a_{22}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} =$$
$$= \left(\sqrt{a_{11}}\frac{\partial\xi}{\partial x} \pm \sqrt{a_{22}}\frac{\partial\xi}{\partial y}\right)\left(\sqrt{a_{11}}\frac{\partial\eta}{\partial x} \pm \sqrt{a_{22}}\frac{\partial\eta}{\partial y}\right),$$

such that, taking into account the relation (6.1.18), we obtain that $\overline{a_{12}} = 0$. ■

Using the fact that $\overline{a_{11}} = \overline{a_{12}} = 0$, we deduce that the parabolical equation has the canonical form

$$\overline{a_{22}}\frac{\partial^2 u}{\partial\eta^2} = F\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right),$$

or, equivalently,

$$\frac{\partial^2 u}{\partial\eta^2} = G\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right).$$

Remark. If, instead of the transformation (6.1.18), we take the transformation

$$\xi = \xi(x, y),$$
$$\eta = \varphi(x, y),$$

where $\varphi(x, y) = C$ is the only one prime integral of the characteristic equation (6.1.11), and $\xi(x, y)$ is an arbitrary function of the class C^2 and which together with $\varphi(x, y)$ assures the fact that the transformation is non-degenerate (that is, $\xi(x, y)$ and $\varphi(x, y)$ satisfies a condition which is analogous to (6.1.19)), then after analogous calculations as in the Proposition 6.1.2, we obtain the following canonical form

$$\frac{\partial^2 u}{\partial \xi^2} = H\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

3^o **The elliptical case**: $\Delta = a_{12}^2 - a_{11}a_{22} < 0$. In this case the characteristic equation (6.1.11) admits two prime integrals, which are complex conjugated and which can be written in the form

$$\varphi(x, y) = C_1,$$
$$\overline{\varphi}(x, y) = C_2,$$

where C_1 and C_2 are arbitrary constants. Also, we have denoted by $\overline{\varphi}$ the function which is a complex conjugated function of the function φ. We proceed as in the hyperbolical case, that is, we take the new variables ξ and η of the form

$$\xi = \varphi(x, y),$$
$$\eta = \overline{\varphi}(x, y),$$

with the condition

$$\left|\frac{\partial(\varphi, \overline{\varphi})}{\partial(x, y)}\right| = \frac{\partial \varphi}{\partial x}\frac{\partial \overline{\varphi}}{\partial y} - \frac{\partial \overline{\varphi}}{\partial x}\frac{\partial \varphi}{\partial y} \neq 0.$$

Then we will obtain $\overline{a_{11}} = \overline{a_{22}} = 0$ and therefore the elliptical equation has the following canonical form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

Unlike to the hyperbolical case, the last equation is in the case of complex numbers.

We want to find such a transformation of variables to obtain the canonical form in the set of the real numbers. With this hope in mind, we introduce the functions $\alpha(x, y)$ and $\beta(x, y)$ such that

$$\alpha = Re(\varphi) = \frac{1}{2}\left(\varphi + \overline{\varphi}\right),$$
$$\beta = Im(\varphi) = \frac{1}{2i}\left(\varphi - \overline{\varphi}\right),$$

and the new variables ξ and η are taken of the form

$$\begin{aligned} \xi &= \alpha + i\beta, \\ \eta &= \alpha - i\beta. \end{aligned} \tag{6.1.19}$$

Proposition 6.1.3 *In the case of the elliptical equations we have*

$$\tilde{a}_{11} = \tilde{a}_{22},\ \tilde{a}_{12} = 0,$$

where $\tilde{a}_{ij}$ are the coefficients of the canonical equation obtained by using the transformation (6.1.19).

Proof It is easy to see that ξ is, in fact, $\xi = \varphi(x, y)$ and then $\overline{a_{11}} = 0$. If we take into account (6.1.19), then we have

$$\begin{aligned} 0 = \overline{a_{11}} = a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + a_{22}\left(\frac{\partial \xi}{\partial y}\right)^2 = \\ a_{11}\left(\frac{\partial \alpha}{\partial x} + i\frac{\partial \beta}{\partial x}\right)^2 + a_{22}\left(\frac{\partial \alpha}{\partial y} + i\frac{\partial \beta}{\partial y}\right)^2 + \\ +2a_{12}\left(\frac{\partial \alpha}{\partial x} + i\frac{\partial \beta}{\partial x}\right)\left(\frac{\partial \alpha}{\partial y} + i\frac{\partial \beta}{\partial y}\right) = \\ = a_{11}\left(\frac{\partial \alpha}{\partial x}\right)^2 + 2a_{12}\frac{\partial \alpha}{\partial x}\frac{\partial \alpha}{\partial y} + a_{22}\left(\frac{\partial \alpha}{\partial y}\right) - \\ -\left[a_{11}\left(\frac{\partial \beta}{\partial x}\right)^2 + 2a_{12}\frac{\partial \beta}{\partial x}\frac{\partial \beta}{\partial y} + a_{22}\left(\frac{\partial \beta}{\partial y}\right)^2\right] + \\ +2i\left[a_{11}\frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial x} + a_{12}\left(\frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial \beta}{\partial x}\right) + a_{22}\frac{\partial \alpha}{\partial y}\frac{\partial}{\partial y}\right]. \end{aligned}$$

This is an equality in the set of the complex numbers and then both the real and imaginary part are null, whence it follows the result from proposition. ■

Using the results from the Proposition 6.1.3, we deduce that in the elliptical case the canonical form of the equation is

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = H\left(\alpha, \beta, u, \frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}\right),$$

where H is a real function.

We can conclude that, in the domain of ellipticity of the Eq. (6.1.3) does not exist a characteristic direction. In the domain of hyperbolicity of the Eq. (6.1.3), in each point there exists two real distinct characteristic directions and in each point of the domain of parabolicity there exists only one real characteristic direction.

As a consequence, if the coefficients a_{11}, a_{12} and a_{22} of the Eq. (6.1.3) are sufficient regular, the domain of hyperbolicity is a network of two families of characteristic curves, and the domain of parabolicity is covered by only one such a family.

As an exemplification, let us consider the equation:

$$y^m \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where m is a natural odd number. In this case the Eq. (6.1.12) receives the form:

$$y^m \left(\frac{dy}{\mathrm{d}x}\right)^2 + 1 = 0.$$

It is easy to see that it does not exist any characteristic direction in the semi-plane $y > 0$. But, in each point of the straight line $y = 0$ and in each point of the semi-plane $y < 0$, there exists a characteristic direction, respectively, two characteristic directions.

We write the equation of the characteristic curves in the form:

$$\mathrm{d}x \pm (-y)^{\frac{m}{2}} \mathrm{d}y = 0,$$

from where, by integration, we deduce that the semi-plane $y < 0$ is covered by two families of real characteristic curves, described by the equations:

$$x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = C_1,$$

and

$$x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = C_2,$$

where C_1 and C_2 are real constants.

6.2 Canonical Form for $n > 2$

In this paragraph we make some considerations on the canonical form of a partial differential equation of the order two, in the case $n > 2$.

Let Ω be an open set from $\mathbb{R}^n$ and consider the quasilinear equation with partial derivatives of the order two

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x) = \frac{\partial^2 u}{\partial x_i \partial x_j} = f\left(x_1, x_2, \ldots, x_n, , u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right), \tag{6.2.1}$$

where $x = (x_1, x_2, \ldots, x_n),\ a_{ij} = a_{ji}(x) \in C(\Omega)$.

The function $f = f(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n)$ is defined and is continuous for any $(x_1, x_2, \ldots, x_n) \in \Omega$ and $-\infty < z, p_1, p_2, \ldots, p_n < \infty$, and u is the unknown function, $u : \Omega \to \mathbb{R},\ u \in C^2(\Omega)$.

We intend to make a change of variables such that in the new equation (which is called the canonical equation) some new coefficients, denoted by $\overline{a_{ij}}$, as in the case $n = 2$, to be null.

Consider the transformation

$$\begin{aligned}
\xi_1 &= \xi_1(x_1, x_2, \ldots, x_n),\\
\xi_2 &= \xi_2(x_1, x_2, \ldots, x_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\xi_n &= \xi_n(x_1, x_2, \ldots, x_n),
\end{aligned} \tag{6.2.2}$$

with the condition

$$\left|\frac{\partial \xi}{\partial x}\right| = \begin{vmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} & \ldots & \frac{\partial \xi_1}{\partial x_n} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} & \ldots & \frac{\partial \xi_2}{\partial x_n} \\ \\ \frac{\partial \xi_n}{\partial x_1} & \frac{\partial \xi_n}{\partial x_2} & \ldots & \frac{\partial \xi_n}{\partial x_n} \end{vmatrix} \neq 0, \tag{6.2.3}$$

where $\xi = \xi(x)$ is a vectorial function, $\xi : \Omega \to \mathbb{R}^n,\ \xi \in C^2(\Omega)$.

Due to condition (6.2.3), based on the theory of implicit functions, the system (6.2.2) can be solved in the vectorial variable x:

$$\begin{aligned}
x_1 &= x_1(\xi_1, \xi_2, \ldots, \xi_n),\\
x_2 &= x_2(\xi_1, \xi_2, \ldots, \xi_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_n &= x_n(\xi_1, \xi_2, \ldots, \xi_n),
\end{aligned}$$

such that at the end, the solution of the Eq. (6.2.1) will be obtained as a function of x. From Eq. (6.2.2) we have

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i},\ \ i = 1, 2, \ldots, n$$

and then

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k=1}^{n} \sum_{m=1}^{n} \frac{\partial^2 u}{\partial \xi_k \partial \xi_m} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_m}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k} \frac{\partial^2 \xi_k}{\partial x_i \partial x_j}. \tag{6.2.4}$$

We introduce Eq. (6.2.4) in (6.2.1) whence it follows the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} a_{ij} \frac{\partial^2 u}{\partial \xi_k \partial \xi_m} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_m}{\partial x_j} = G\left(\xi, u, \frac{\partial u}{\partial \xi_1}, \frac{\partial u}{\partial \xi_2}, \ldots, \frac{\partial u}{\partial \xi_n}\right). \tag{6.2.5}$$

We introduce the notation

$$\overline{a_{km}}(\xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_m}{\partial x_j} \tag{6.2.6}$$

and then the Eq. (6.2.5) becomes

$$\sum_{k=1}^{n} \sum_{m=1}^{n} \overline{a_{km}}(\xi) \frac{\partial^2 u}{\partial \xi_k \partial \xi_m} = G\left(\xi, u, \frac{\partial u}{\partial \xi_1}, \frac{\partial u}{\partial \xi_2}, \ldots, \frac{\partial u}{\partial \xi_n}\right). \tag{6.2.7}$$

fix $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \Omega$ and use the notation

$$\lambda_{ik} = \frac{\partial \xi_k}{\partial x_i}(x^0)$$

such that from Eq. (6.2.6) we deduce

$$\overline{a_{km}}(\xi^0) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x^0) \lambda_{ik} \lambda_{jm} \tag{6.2.8}$$

where $\xi^0 = \xi(x^0)$.

Using a matrix notation

$$\overline{A} = [\overline{a_{km}}], \ A = \left[a_{ij}\right], \ \Lambda = \left[\lambda_{ij}\right],$$

the Eq. (6.2.8) becomes

$$\overline{A} = \Lambda^t A \Lambda, \tag{6.2.9}$$

where we have denoted by Λ^t the transposed of the matrix Λ.

It is well known that if, in Eq. (6.2.9), we make the change of variables

$$\Lambda = TM,$$

where by M we have denoted a non-degenerate matrix, and by T an orthogonal matrix ($T^t = T^{-1}$), then the matrix $\overline{A}$ reduces to the its diagonal form, that is, the matrix which has nonzero elements only on the principal diagonal.

With regard to the elements on the principal diagonal, we have the following Sylvester's rule of inertness:

The number of the positive elements on the principal diagonal is constant. Also, the number of the negative elements on the principal diagonal is constant. We have the following variants:

1^o If all elements on the principal diagonal are strict positive in a point $\xi^0 \in \Omega$, then the canonical equation becomes

$$\sum_{j=1}^{n} \overline{a_{jj}} \frac{\partial^2 u}{\partial \xi_j^2}(\xi^0) = G\left(\xi_1, \xi_2, \ldots, \xi_n, u, \frac{\partial u}{\partial \xi_1}, \frac{\partial u}{\partial \xi_2}, \ldots, \frac{\partial u}{\partial \xi_n}\right).$$

Then, we say that the quasi-linear partial differential equation of the order two is *elliptical* in the point $\xi^0 \in \Omega$.

2^o If does not exist a zero on the principal diagonal, but there exist both positive elements and negative elements, then we say that the equation is *hyperbolical* in the point $\xi^0 \in \Omega$. In the particular case in which only one element is strictly positive and all others are strictly negative, we say that the equation is *ultrahyperbolical* in the respective point.

3^o If on the principal diagonal there exist certain null elements, then the equation is *parabolical* in the respective point.

4^o If on the principal diagonal there exist both null elements and non-zero elements and these have all the same sign then the equation is *elliptical-parabolical* in the respective point.

5^o If on the principal diagonal there exist null elements and non-zero elements having different signs then we say that the equation is *hyperbolical-parabolical*.

It is clear that the benefit for the canonical form of a quasi-linear equation with partial derivatives of order two is given by the fact that this form of the equation facilitates its integration.

Chapter 7
Hyperbolical Equations

7.1 Problem of the Infinite Vibrating Chord

The main representative of hyperbolical equations is considered to be the equation of the vibrating chord, also called the equation of waves.

Firstly, we consider the case of the infinite chord. Properly, the chord is not infinite, but its length is much more than its cross section. The general aim of this paragraph is to study the following initial-boundary values problem, attached to the equation of the infinite chord

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x),\ \forall (t,x) \in (-\infty, +\infty),\ \forall t > 0,\\
&u(0,x) = \varphi(x),\ \forall x \in (-\infty, +\infty),\\
&\frac{\partial u}{\partial t}(0,x) = \psi(x),\ \forall x \in (-\infty, +\infty),
\end{aligned}
\tag{7.1.1}
$$

where the functions $f(t,x)$, $\varphi(x)$ and $\psi(x)$ are given and continuous on their domain of definition. The function $u = u(t,x)$ is the unknown function of the problem and represents the amplitude of the chord at the moment t, at the point x. The positive constant a is prescribed for each type of the material of the chord.

We will decompose the Cauchy's problem (7.1.1) in two other problems, one homogeneous with regard to the right-hand side of the equation and, second, homogeneous with regard to the initial conditions:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0,\ \forall (t,x) \in (-\infty, +\infty),\ \forall t > 0,\\
&u(0,x) = \varphi(x),\ \forall x \in (-\infty, +\infty),\\
&\frac{\partial u}{\partial t}(0,x) = \psi(x),\ \forall x \in (-\infty, +\infty),
\end{aligned}
\tag{7.1.2}
$$

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and, respectively,

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x),\ \forall (t, x) \in (-\infty, +\infty),\ \forall t > 0, \\ &\qquad u(0, x) = 0,\ \forall x \in (-\infty, +\infty), \\ &\qquad \frac{\partial u}{\partial t}(0, x) = 0,\ \forall x \in (-\infty, +\infty). \end{aligned} \tag{7.1.3}$$

Proposition 7.1.1 *If the function $u_1(t, x)$ is a solution of the problem (7.1.2) and the function $u_2(t, x)$ is a solution of the problem (7.1.3), then the function*

$$u(t, x) = u_1(t, x) + u_2(t, x),\ \forall (t, x) \in (-\infty, +\infty),\ \forall t > 0, \tag{7.1.4}$$

is a solution of the problem (7.1.1).

Proof Firstly, we verify the initial conditions:

$$\begin{aligned} u(0, x) &= u_1(0, x) + u_2(0, x) = \varphi(x) + 0 = \varphi(x), \\ \frac{\partial u}{\partial t}(0, x) &= \frac{\partial u_1}{\partial t}(0, x) + \frac{\partial u_2}{\partial t}(0, x) = \psi(x) + 0 = \psi(x), \end{aligned}$$

where we haven taken into account the initial conditions $(7.1.2)_2$ and $(7.1.3)_2$, respectively, $(7.1.2)_3$ and $(7.1.3)_3$.

By using the linearity of the derivative, by derivation in (7.1.4), we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial^2 u_1}{\partial t^2} - a^2 \frac{\partial^2 u_1}{\partial x^2} \right) + \left(\frac{\partial^2 u_2}{\partial t^2} - a^2 \frac{\partial^2 u_2}{\partial x^2} \right) = \\ &= 0 + f(t, x) = f(t, x), \end{aligned}$$

where we haven taken into account Eqs. $(7.1.2)_1$ and $(7.1.3)_1$. ∎

Now, let us solve the problems (7.1.2) and (7.1.3) and then based on the Proposition 7.1.1, we obtain the solution of the problem (7.1.1).

With regard to the problem (7.1.3) we have the following result.

Theorem 7.1.1 *The function $U(t, x)$ defined by*

$$U(t, x) = \frac{1}{2a} \int_0^t \left\{ \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau, \xi) \mathrm{d}\xi \right\} \mathrm{d}\tau, \tag{7.1.5}$$

is the solution for the problem (7.1.3).

Proof It is clear that

$$U(0, x) = \frac{1}{2a} \int_0^0 \left\{ \int_{x-a(0-\tau)}^{x+a(0-\tau)} f(\tau, \xi) \mathrm{d}\xi \right\} \mathrm{d}\tau = 0.$$

Therefore, by using the rule of derivation of an integral with parameter, it results

$$\frac{\partial U(t,x)}{\partial t} = \frac{1}{2a}\int_x^x f(t,\xi)\mathrm{d}\xi + \frac{1}{2a}\int_0^t \frac{\partial}{\partial t}\left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau =$$
$$= \frac{1}{2a}\int_0^t a[f(\tau,x+a(t-\tau)) + f(\tau,x-a(t-\tau))]\mathrm{d}\tau +$$
$$+ \frac{1}{2a}\int_0^t \left[\int_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\partial f(\tau,\xi)}{\partial t}\mathrm{d}\xi\right]\mathrm{d}\tau =$$
$$= \frac{1}{2}\int_0^t [f(\tau,x+a(t-\tau)) + f(\tau,x-a(t-\tau))]\mathrm{d}\tau.$$

Now, it is easy to see that

$$\frac{\partial U}{\partial t}(0,x) = \frac{1}{2}\int_0^0 [f(\tau,x+a(0-\tau)) + f(\tau,x-a(0-\tau))]\mathrm{d}\tau = 0.$$

We use the derivative on the previous relation again with respect to t:

$$\frac{\partial^2 U}{\partial t^2}(t,x) = \frac{1}{2}[f(t,x+a.0) + f(t,x-a.0)] +$$
$$+ \frac{1}{2}\int_0^t \frac{\partial}{\partial t}[f(\tau,x+a(t-\tau)) + f(\tau,x-a(t-\tau))]\,\mathrm{d}\tau,$$

that is,

$$\frac{\partial^2 U}{\partial t^2}(t,x) = f(t,x) + \frac{a}{2}\int_0^t \left[\frac{\partial f(\tau,x+a(t-\tau))}{\partial (x+a(t-\tau))} - \frac{\partial f(\tau,x-a(t-\tau))}{\partial (x-a(t-\tau))}\right]\mathrm{d}\tau. \tag{7.1.6}$$

By derivation with respect to x, by using the rule of derivation of an integral with parameter, it results

$$\frac{\partial U(t,x)}{\partial x} = \frac{1}{2a}\int_0^t \frac{\partial}{\partial x}\left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau =$$
$$= \frac{1}{2a}\int_0^t [f(\tau,x+a(t-\tau)) - f(\tau,x-a(t-\tau))]\mathrm{d}\tau.$$

Here, we derive one more time, again with respect to x, such that we are led to

$$\frac{\partial^2 U(t,x)}{\partial x^2} = \frac{1}{2a}\int_0^t \frac{\partial}{\partial x}[f(\tau,x+a(t-\tau)) - f(\tau,x-a(t-\tau))]\,\mathrm{d}\tau =$$

$$= \frac{1}{2a}\int_0^t \left[\frac{\partial f(\tau, x+a(t-\tau))}{\partial(x+a(t-\tau))}\frac{\partial(x+a(t-\tau))}{\partial x} - \right.$$
$$\left. - \frac{\partial f(\tau, x-a(t-\tau))}{\partial(x-a(t-\tau))}\frac{\partial(x-a(t-\tau))}{\partial x}\right]\mathrm{d}\tau.$$

Therefore

$$\frac{\partial^2 U(t,x)}{\partial x^2} = \frac{1}{2a}\int_0^t \left[\frac{\partial f(\tau, x+a(t-\tau))}{\partial(x+a(t-\tau))} - \frac{\partial f(\tau, x-a(t-\tau))}{\partial(x-a(t-\tau))}\right]\mathrm{d}\tau, \quad (7.1.7)$$

such that, from Eqs. (7.1.6) and (7.1.7), we obtain

$$\frac{\partial^2 U(t,x)}{\partial t^2} - a^2\frac{\partial^2 U(t,x)}{\partial x^2} = f(t,x),$$

that is, $U(t,x)$ verifies the Eq. $(7.1.3)_1$. ∎

Now, let us solve the problem (7.1.2).

Theorem 7.1.2 *The solution of the problem (7.1.2) is given by*

$$u(t,x) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\mathrm{d}s.$$

Proof As a first step, we obtain the canonical form of Eq. $(7.1.2)_1$.
By using the considerations from the Sect. 1.1 (the Chap. I), the characteristic equation in our case is

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - a^2 = 0.$$

We can observe that $\Delta = a^2 > 0$ and, therefore, we are, indeed, in the case of the hyperbolical equations. One immediately obtains the prime integrals

$$x + at = C_1,$$
$$x - at = C_2,$$

where C_1 and C_2 are arbitrarily constants. Then we perform the change of variables

$$\xi = x + at,$$
$$\eta = x - at. \quad (7.1.8)$$

It is easy to see that the transformation (7.1.8) is non-singular, because

$$\left|\frac{\partial(\xi,\eta)}{\partial(t,x)}\right| = \begin{vmatrix} a & -a \\ 1 & 1 \end{vmatrix} = 2a > 0.$$

With the change of variables (7.1.8), the canonical form is

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

that is,

$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0 \Rightarrow \frac{\partial u}{\partial \xi} = \gamma(\xi),\ \ \gamma \in C^1((-\infty, +\infty)).$$

After one more integration, we obtain

$$u(\xi, \eta) = \int \gamma(\xi)\mathrm{d}\xi + \beta(\eta) = \alpha(\xi) + \beta(\eta), \tag{7.1.9}$$

where α is an antiderivative of the arbitrary function γ.

If we suppose that α and β are functions of the class C^1, then the order of the above derivation has no importance, according to the Schwarz's classical criterion. But, in order to verify the equation with partial derivatives of order two, the functions α and β must be functions of the class C^2.

Introducing Eq. (7.1.8) into (7.1.9), it follows

$$u(t, x) = \alpha(x + at) + \beta(x - at), \tag{7.1.10}$$

where the functions α and β will be determined with the aid of the initial conditions:

$$\varphi(x) = u(0, x) = \alpha(x) + \beta(x),$$
$$\psi(x) = \frac{\partial u}{\partial t}(0, x) = a\alpha'(x) - a\beta'(x).$$

This system is equivalent to

$$\alpha(x) + \beta(x) = \varphi(x),$$
$$\alpha(x) - \beta(x) = \frac{1}{a}\int_0^x \psi(s)\mathrm{d}s + C,$$

where C is an arbitrary constant of integration. The solution of this system is

$$\alpha(x) = \frac{\varphi(x)}{2} + \frac{1}{2a}\int_0^x \psi(s)\mathrm{d}s + \frac{C}{2},$$
$$\beta(x) = \frac{\varphi(x)}{2} - \frac{1}{2a}\int_0^x \psi(s)\mathrm{d}s - \frac{C}{2},$$

such that, from Eq. (7.1.10), we obtain

$$u(t,x)=\frac{\varphi(x+at)}{2}+\frac{1}{2a}\int_0^{x+at}\psi(s)\mathrm{d}s+\frac{C}{2}+$$
$$+\frac{\varphi(x-at)}{2}-\frac{1}{2a}\int_0^{x-at}\psi(s)\mathrm{d}s-\frac{C}{2}=$$
$$=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\mathrm{d}s.$$

This is even the desired result and the theorem is proved. ■

Remark. Based on the results from the Theorem 7.1.1, the Theorem 7.1.2 and the Proposition 7.1.1, we will deduce that the solution of the problem (7.1.1) is

$$u(t,x)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\mathrm{d}s+$$
$$+\frac{1}{2a}\int_0^t\left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau. \tag{7.1.11}$$

In this way, we proved the following result of existence.

Theorem 7.1.3 (of existence) *If the given function $f(t,x)$ is assumed to be of the class $C^0((0,\infty)\times(-\infty,+\infty))$, the given function $\varphi(x)$ is of the class $C^2(-\infty,+\infty)$ and the given function $\psi(x)$ is of the class $C^1(-\infty,+\infty)$, then the nonhomogeneous problem of the infinite chord admits the classical solution (7.1.11).*

We call *a classical solution* a function $u=u(t,x)$ of the class C^2 with respect to $x\in(-\infty,+\infty)$ and $t>0$, which verifies the initial conditions $(7.1.1)_2$ and $(7.1.1)_3$ and if it is replaced in the Eq. $(7.1.1)_1$, we obtain an identity.

Remark. The form (7.1.11) of the solution of the problem (7.1.10) is also called *D'Alembert's formula* for the nonhomogeneous problem of the infinite chord.

In the following theorem we prove the uniqueness of the solution of the Cauchy's problem (7.1.1).

Theorem 7.1.4 (of uniqueness) *The single classical solution of the nonhomogeneous problem of the infinite chord is that given in (7.1.11).*

Proof We suppose, through absurd, that the problem (7.1.1) admits two classical solutions $u_1(t,x)$ and $u_2(t,x)$ and then

$$\begin{aligned}&\frac{\partial^2 u_i}{\partial t^2}-a^2\frac{\partial^2 u_i}{\partial x^2}=f(t,x),\ \forall(t,x)\in(-\infty,+\infty),\ \forall t>0,\\&u_i(0,x)=\varphi(x),\ \forall x\in(-\infty,+\infty),\\&\frac{\partial u_i}{\partial t}(0,x)=\psi(x),\ \forall x\in(-\infty,+\infty),\end{aligned} \tag{7.1.12}$$

where $i = 1, 2$. We define the function $v(t, x)$ by

$$v(t, x) = u_1(t, x) - u_2(t, x).$$

Then

$$\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2} - a^2 \frac{\partial^2 u_1}{\partial x^2} = \left(\frac{\partial^2 u_2}{\partial t^2} - a^2 \frac{\partial^2 u_2}{\partial x^2} \right) =$$
$$= f(t, x) - f(t, x) = 0,$$

where we have used Eq. $(7.1.12)_1$.

Therefore,

$$v(0, x) = u_1(0, x) - u_2(0, x) = \varphi(x) - \varphi(x) = 0,$$
$$\frac{\partial v}{\partial t}(0, x) = \frac{\partial u_1}{\partial t}(0, x) - \frac{\partial u_2}{\partial t}(0, x) = \psi(x) - \psi(x) = 0,$$

where we have used the initial conditions $(7.1.12)_2$ and $(7.1.12)_3$. Thus, the function v satisfies a problem of the form (7.1.1), where $f(t, x) = \varphi(x) = \psi(x) = 0$. Then, according to Eq. (7.1.11), we have

$$v(t, x) = 0 \Rightarrow u_1(t, x) = u_2(t, x),$$

that concludes the proof of the theorem. ■

In order to obtain a result of stability with regard to "the right-hand side" and initial conditions, for the problem (7.1.1), we consider that $t \in (0, T]$, where T suitably chosen moment.

Theorem 7.1.5 (of stability) *We denote by $u_1(t, x)$, respectively $u_2(t, x)$, the solutions (unique) of the following two problems*

$$\begin{aligned} &\frac{\partial^2 u_i}{\partial t^2} - a^2 \frac{\partial^2 u_i}{\partial x^2} = f_i(t, x), \ \forall (t, x) \in (-\infty, +\infty), \ \forall t > 0, \\ &u_i(0, x) = \varphi_i(x), \ \forall x \in (-\infty, +\infty), \\ &\frac{\partial u_i}{\partial t}(0, x) = \psi_i(x), \ \forall x \in (-\infty, +\infty), \end{aligned} \tag{7.1.13}$$

where $i = 1, 2$ and T is a fixed moment, in a way that we will see below. Then for any $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that if

$$\begin{aligned} |f(t, x)| &= |f_1(t, x) - f_2(t, x)| < \delta, \\ |\varphi(t, x)| &= |\varphi_1(t, x) - \varphi_2(t, x)| < \delta, \\ |\psi(t, x)| &= |\psi_1(t, x) - \psi_2(t, x)| < \delta, \end{aligned} \tag{7.1.14}$$

then

$$|u(t,x)| = |u_1(t,x) - u_2(t,x)| < \varepsilon.$$

Proof Based on the Theorems 7.1.3 and 7.1.4, the single classical solutions of the problems (7.1.13) are the functions $u_i(t,x)$, given by

$$u_i(t,x) = \frac{1}{2}[\varphi_i(x+at) + \varphi_i(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi_i(s)\mathrm{d}s + \\ + \frac{1}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f_i(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau,$$

where $i = 1, 2$. Now, we make the difference of these two solutions

$$u_1(t,x) - u_2(t,x) = \frac{1}{2}\left[\varphi_1(x+at) - \varphi_1(x+at)\right] + \\ + \frac{1}{2}\left[\varphi_1(x-at) - \varphi_2(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\left[\psi_1(s) - \psi_2(s)\right]\mathrm{d}s + \\ + \frac{1}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} \left[f_1(\tau,\xi) - f_2(\tau,\xi)\right]\mathrm{d}\xi\right\}\mathrm{d}\tau.$$

If we take the modulus in this equality and use the inequality of the triangle, we obtain that the modulus of the right-hand side is less than the sum. We use then the fact that the modulus of an integral is less than the integral of the modulus:

$$|u_1(t,x) - u_2(t,x)| \le \frac{1}{2}\left|\varphi_1(x+at) - \varphi_1(x+at)\right| + \\ + \frac{1}{2}\left|\varphi_1(x-at) - \varphi_2(x-at)\right| + \frac{1}{2a}\int_{x-at}^{x+at}\left|\psi_1(s) - \psi_2(s)\right|\mathrm{d}s + \\ + \frac{1}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} \left|f_1(\tau,\xi) - f_2(\tau,\xi)\right|\mathrm{d}\xi\right\}\mathrm{d}\tau.$$

If we take into account Eq. (7.1.14), this inequality leads to

$$|u_1(t,x) - u_2(t,x)| \le \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{2a}\int_{x-at}^{x+at}\mathrm{d}s + \\ + \frac{\delta}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)}\mathrm{d}\xi\right\}\mathrm{d}\tau = \delta + \delta t + \frac{\delta}{2a}\int_0^t 2a(t-\tau)\mathrm{d}\tau = \\ = \delta\left(1 + t + \frac{t^2}{2}\right) \le \delta\left(1 + T + \frac{T^2}{2}\right).$$

If we choose T such that

$$1 + T + \frac{T^2}{2} < \frac{\varepsilon}{\delta},$$

we obtain

$$|u_1(t, x) - u_2(t, x)| < \varepsilon,$$

that concludes the proof of the theorem. ∎

At the end of this paragraph we make some comments with regard to the results of existence, uniqueness and stability from the previous Theorems 7.1.3, 7.1.4 and 7.1.5.

In the case of a problem with initial data, or with boundary data, or, more generally, in the case of a mixed initial-boundary values problem, there exists the concept of the *well posed*, used for the first time by Hadamard. This concept means that for the corresponding problem we have a theorem of uniqueness of the solution.

A theorem of uniqueness can be proved only for certain classes of functions. In the case of the problem of the infinite chord, previously exposed we cannot have a classical solution if we do not suppose that the prescribed functions f, φ and ψ are continuous. Therefore, the class of the continuous functions is the class where one can put the problem of the uniqueness of the solution.

If we want to prove only the uniqueness of the solution, then it is sufficient to suppose that the functions f, φ and ψ are of the class C^0 on their domains of definition. If we want to prove the existence of the solution, it is necessary to suppose that the functions φ and ψ are of the class C^1.

So, it appears the concept of the class of correctness for initial and boundary conditions. This is the class of the functions where it must be considered the functions from the initial and boundary conditions such that we have the uniqueness of the solution for the respective problem.

After we prove the theorem of existence and the theorem of uniqueness, we can talk about the existence and the uniqueness of the solutions of the problems for which the function "right-hand side" is given and, also, the functions from the initial and boundary conditions are prescribed.

A *particular solution* of a given problem is the solution that uniquely corresponds (by virtue of the theorem of existence and uniqueness) to the right-hand side, to some boundary data and to some fixed initial data. Therefore, for each fixed right-hand side, initial data and boundary data we have a particular solution. In this context, *the general solution* will be the family of all particular solutions.

In some cases, there exists certain solutions for which we cannot prove the theorem of existence and uniqueness. This solution is called *a singular solution*.

The functions which define the right-hand side, the initial conditions and the boundary conditions are given by the experiment. In the case of the problem of the vibrating infinite chord, for the functions f_1, φ_1 and ψ_1 given by an experimentalist, we have an unique determined solution u_1.

If another experimentalist delivers the data f_2, φ_2 and ψ_2, for the same phenomenon, the problem will admit the unique determined solution u_2.

If the data f_1, φ_1 and ψ_1 is sufficient little different from the data f_2, φ_2 and ψ_2 we have that the corresponding solutions u_1 and, respectively u_2, are sufficient near, we say that *the solution is stable*.

7.2 Initial-Boundary Values Problems

Let Ω be a bounded domain from the space $\mathbb{R}^n$ with the boundary $\partial\Omega$ having a tangent plane, piecewise continuously varying. As usual, we note by $\mathcal{T}_T$ the temporal interval $\mathcal{T}_T = (0, T]$ and $\overline{\mathcal{T}_T} = [0, T]$, where $T > 0$.

Consider the initial boundary values, attached to the equation of the waves

$$\begin{aligned}
\Delta u(t,x) - u_{tt}(t,x) &= f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \Omega,\\
u(t,y) &= \alpha(t,y),\ \forall x \in \overline{\mathcal{T}_T} \times \partial\Omega,\\
u(0,x) &= \varphi(x),\ \forall x \in \overline{\Omega},\\
u_t(0,x) &= \psi(x),\ \forall x \in \overline{\Omega},
\end{aligned} \tag{7.2.1}$$

where the functions $f,\ \alpha,\ \varphi$ and ψ are given and continuous on their domains of definition.

Definition 7.2.1 We call the classical solution of the problem (7.2.1), the function $u = u(t,x)$ which satisfies the conditions:

- u is a continuous function on $\overline{\mathcal{T}_T} \times \overline{\Omega}$;
- the derivatives $u_{x_i x_i}$ and u_{tt} are continuous functions on $\mathcal{T}_T \times \Omega$;
- u satisfies Eq. $(7.2.1)_1$, the boundary condition $(7.2.1)_2$ and the initial conditions $(7.2.1)_3$ and $(7.2.1)_4$.

We use a power method to show that the problem (7.2.1) has only one solution.

Theorem 7.2.1 *The initial-boundary values problem (7.2.1) has at the most one classical solution.*

Proof We suppose, ad absurdum, that the problem (7.2.1) has two classical solutions, $u_1(t,x)$ and $u_2(t,x)$. We define the function v by

$$v(t,x) = u_1(t,x) - u_2(t,x).$$

It is easy to see that the function v satisfies the conditions imposed to a classical solutions, since $u_1(t,x)$ and $u_2(t,x)$ are classical solutions. Also, v satisfies the problem (7.2.1) in its homogeneous form

$$\begin{aligned}
&\Delta v(t,x) - v_{tt}(t,x) = 0,\ \forall (t,x) \in \mathcal{T}_T \times \Omega, \\
&v(t,y) = 0,\ \forall (t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega, \\
&v(0,x) = 0,\ \forall x \in \overline{\Omega}, \\
&v_t(0,x) = 0,\ \forall x \in \overline{\Omega}.
\end{aligned} \tag{7.2.2}$$

Now, we attach to the function v, the function E defined by

$$E(t) = \frac{1}{2} \int_\Omega \left[v_t^2(t,\xi) + \sum_{i=1}^n v_{x_i}^2(t,\xi) \right] \mathrm{d}\xi, \tag{7.2.3}$$

which is called *the power integral*.

We will give the proof in two steps. In the first step we will show that $E(0) = 0$, and, in the second step, we prove that

$$\frac{dE(t)}{\mathrm{d}t} = 0,$$

whence it follows the conclusion that, in fact, $E(t)$ is a constant. But, according to the first step, $E(0) = 0$, and then the conclusion will be that $E \equiv 0$. This conclusion together with the definition (7.2.3) of the function E lead to the conclusion that

$$v_t = 0,\ \ v_{x_i} = 0,\ \ i = 1, 2, ..., n,$$

which proves that v is a constant. But, on the boundary, the function v is zero and, therefore, we will deduce that this constant is null, that is, $v \equiv 0$ such that $u_1 \equiv u_2$.

The first step can be immediately proved. We direct substitute $t = 0$ and obtain

$$E(0) = \frac{1}{2} \int_\Omega \left[v_t^2(0,\xi) + \sum_{i=1}^n v_{x_i}^2(0,\xi) \right] \mathrm{d}\xi = 0,$$

where we have used the initial conditions $(7.2.2)_3$ and $(7.2.2)_4$.

Now, we approach the second step. Due to the conditions of regularity is satisfied by the function v, we can derive in Eq. (7.2.3) under the integral

$$\frac{dE(t)}{\mathrm{d}t} = \int_\Omega \left[v_t(t,\xi) v_{tt}(t,\xi) + \sum_{i=1}^n v_{x_i}(t,\xi) v_{tx_i}(t,\xi) \right] \mathrm{d}\xi. \tag{7.2.4}$$

But

$$\begin{aligned}
&\int_\Omega v_{x_i}(t,\xi) v_{tx_i}(t,\xi) \mathrm{d}\xi = \int_\Omega \frac{\partial}{\partial x_i} \left[v_{x_i}(t,\xi) v_t(t,\xi) \right] \mathrm{d}\xi - \\
&- \int_\Omega v_t(t,\xi) v_{x_i x_i}(t,\xi) \mathrm{d}\xi = \int_{\partial\Omega} v_{x_i}(t,\xi) v_t(t,\xi) \cos\alpha_i \mathrm{d}\sigma_\xi -
\end{aligned}$$

$$-\int_{\Omega} v_t(t,\xi)v_{x_i x_i}(t,\xi)\mathrm{d}\xi = -\int_{\Omega} v_t(t,\xi)v_{x_i x_i}(t,\xi)\mathrm{d}\xi, \qquad (7.2.5)$$

where, firstly, we have used the Gauss–Ostrogradski's formula (that has been possible, taking into account that the surface $\partial\Omega$ admits tangent plane).

Then we have used the boundary condition (7.2.2)$_2$.

It results from Eq. (7.2.5) that

$$\int_{\Omega} \sum_{i=1}^{n} v_{x_i}(t,\xi)v_{x_i x_i}(t,\xi)\mathrm{d}\xi = -\int_{\Omega} v_t(t,\xi)\Delta v(t,\xi)\mathrm{d}\xi,$$

and then Eq. (7.2.4) becomes

$$\frac{dE(t)}{\mathrm{d}t} = \int_{\Omega} v_t(t,\xi)\left[v_{tt}(t,\xi) - \Delta v(t,\xi)\right]\mathrm{d}\xi = 0,$$

since v satisfies the homogeneous equation (7.2.2)$_1$. ■

We will prove now a result of stability for the solution of the problem (7.2.1), with regard to the right-hand side of the equation and, also, to the initial conditions.

Theorem 7.2.2 *Consider $u_1(t,x)$ and $u_2(t,x)$ are the solutions of the problems*

$$\begin{aligned}
&\Delta u_i(t,x) - \frac{\partial^2 u_i}{\partial t^2}(t,x) = f_i(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \Omega,\\
&u_i(t,y) = \alpha(t,y),\ \forall x \in \overline{\mathcal{T}_T} \times \partial\Omega,\\
&\quad u_i(0,x) = \varphi_i(x),\ \forall x \in \overline{\Omega},\\
&\frac{\partial u_i}{\partial t}(0,x) = \psi_i(x),\ \forall x \in \overline{\Omega},
\end{aligned}$$

where $i = 1, 2$.

We suppose that for $\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0$ such that

$$\begin{aligned}
|f_1(t,x) - f_2(t,x)| &< \delta,\\
|\varphi_1(t,x) - \varphi_2(t,x)| &< \delta,\\
\left|\frac{\partial \varphi_1}{\partial x_i}(t,x) - \frac{\partial \varphi_1}{\partial x_i}(t,x)\right| &< \delta,\\
|\psi_1(t,x) - \psi_2(t,x)| &< \delta.
\end{aligned}$$

Then

$$|u_1(t,x) - u_2(t,x)| < \varepsilon.$$

Proof We denote by $u(t, x)$ the difference of those two solutions

$$u(t, x) = u_1(t, x) - u_2(t, x),$$

and attach the power integral

$$E(t) = \frac{1}{2}\int_\Omega \left[u_t^2(t, \xi) + \sum_{i=1}^n u_{x_i}^2(t, \xi)\right] d\xi. \tag{7.2.6}$$

Due to the conditions of regularity is satisfied by the function u, we can derive under the integral in Eq. (7.2.6)

$$\frac{dE(t)}{dt} = \int_\Omega u_t(t, \xi)\left[u_{tt}(t, \xi) - \Delta u(t, \xi)\right] d\xi + \\ + \int_{\partial\Omega} u_t(t, \xi)\sum_{i=1}^n u_{x_i}(t, \xi)\cos\alpha_i d\sigma_\xi, \tag{7.2.7}$$

after that we have used the Gauss–Ostrogradski's formula, as in the proof of the Theorem 7.2.1. But on the boundary we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i} = \frac{\partial \alpha}{\partial x_i} - \frac{\partial \alpha}{\partial x_i} = 0. \tag{7.2.8}$$

Also,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = \frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 - \frac{\partial^2 u_2}{\partial t^2} + \Delta u_2 = \\ - f_1(t, x) + f_2(t, x).$$

If we denote by

$$f(t, x) = f_1(t, x) - f_2(t, x)$$

and take into account Eq. (7.2.8), the derivative from Eq. (7.2.7) becomes

$$\frac{dE(t)}{dt} = -\int_\Omega u_t(t, \xi) f(t, \xi) d\xi. \tag{7.2.9}$$

It is easy to prove the inequality

$$\pm ab \le \frac{a^2}{2} + \frac{b^2}{2}. \tag{$*$}$$

Thus, from Eq. (7.2.9) we will deduce

$$\frac{dE(t)}{\mathrm{d}t} \leq \frac{1}{2}\int_{\Omega} u_t^2(t,\xi)\mathrm{d}\xi + \frac{1}{2}\int_{\Omega} f^2(t,\xi)\mathrm{d}\xi. \tag{7.2.10}$$

Based on the hypothesis

$$|f(t,x)| = |f_1(t,x) - f_2(t,x)| < \delta$$

we will deduce that the last integral from Eq. (7.2.10) is arbitrarily small. We use the notation

$$A(t) = \frac{1}{2}\int_{\Omega} f^2(t,\xi)\mathrm{d}\xi.$$

Taking into account Eq. (7.2.6), it is clear that

$$\frac{1}{2}\int_{\Omega} u_t^2(t,\xi)\mathrm{d}\xi \leq E(t),$$

and then Eq. (7.2.10) becomes

$$\frac{dE(t)}{\mathrm{d}t} \leq E(t) + A(t), \tag{7.2.11}$$

such that, by multiplying it with e^{-t}, it results

$$\frac{d}{\mathrm{d}t}\left[e^{-t}E(t)\right] \leq A(t)e^{-t}.$$

We integrate on the interval $[0, t]$ and obtain

$$e^{-t}E(t) \leq E(0) + \int_0^t e^{-\tau}A(\tau)\mathrm{d}\tau,$$

and this relation can be written in the form

$$E(t) \leq e^t E(0) + \int_0^t e^{t-\tau}A(\tau)\mathrm{d}\tau.$$

Since $t \in (0, T]$, the last inequality leads to

$$E(t) \leq e^T E(0) + \int_0^T e^{T-\tau}A(\tau)\mathrm{d}\tau. \tag{7.2.12}$$

By using the hypothesis of the theorem, we will deduce that $E(0)$ is arbitrarily small and, since also, $A(t)$ is arbitrarily small too, it results that the integral from Eq. (7.2.12) is arbitrarily small. Therefore, the function $E(t)$ is superior bounded by a constant which can be arbitrarily small. To show that u is arbitrarily small, we define the function $E_1(t)$ by

$$E_1(t) = \frac{1}{2}\int_\Omega u^2(t,\xi)\mathrm{d}\xi. \tag{7.2.13}$$

Based on the hypothesis of regularity of the function u, we can derive under the integral in Eq. (7.2.13) and then obtain

$$\frac{dE_1(t)}{\mathrm{d}t} = \int_\Omega u(t,\xi)u_t(t,\xi)\mathrm{d}\xi \le$$
$$\le \frac{1}{2}\int_\Omega u_t^2(t,\xi)\mathrm{d}\xi + \frac{1}{2}\int_\Omega u^2(t,\xi)\mathrm{d}\xi,$$

after that we have used again the above inequality (*).

We already proved

$$\frac{dE_1(t)}{\mathrm{d}t} \le E_1(t) + E(t),$$

and, using the same procedure as in case of Eq. (7.2.11), it results

$$E_1(t) \le e^T E_1(0) + \int_0^T e^{T-\tau}E(\tau)\mathrm{d}\tau.$$

Since $E_1(0)$ is arbitrarily small, and also, $E(t)$ is arbitrarily small too, we will deduce that $E_1(t)$ is arbitrarily small and then u is arbitrarily small. ■

7.3 Cauchy's Problem

The initial-boundary values problems from the previous paragraph contain the conditions imposed on the surface which enclosed the body where the problem is stated. In this paragraph it is assumed that the surface is to a great distance such that we can consider that the domain of the problem is the whole space. Therefore, the boundary condition from the formulation of the problem disappears.

We will consider the problem with initial data, that is, the Cauchy's problem, in the Euclidian three-dimensional space $\mathbb{R}^3$. Therefore, we have the problem

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t,x,y,z)-a^2\Delta u(t,x,y,z) &= f(t,x,y,z), \forall (t,x,y,z)\in(0,+\infty)\times\mathbb{R}^3,\\
u(0,x,y,z) &= \varphi(x,y,z),\ \forall (x,y,z)\in\mathbb{R}^3, \qquad (7.3.1)\\
\frac{\partial u}{\partial t}(0,x,y,z) &= \psi(x,y,z),\ \forall (x,y,z)\in\mathbb{R}^3,
\end{aligned}$$

where the functions f, φ and ψ are given and continuous on their domains of definition, and a is a positive known constant of material.

We call *a classical solution* of the problem (7.3.1) a function $u = u(t,x,y,z)$ which satisfies the conditions:

- u and its derivatives of the first order are continuous functions on $[0,+\infty)\times\mathbb{R}^3$;
- the homogeneous derivatives of the order two of the function u are continuous functions on $(0,+\infty)\times\mathbb{R}^3$;
- u verifies Eq. $(7.3.1)_1$ and satisfies the conditions $(7.3.1)_2$ and $(7.3.1)_3$.

We define the function $u(t,x,y,z)$ by

$$u(t,x,y,z) = U_f(t,x,y,z) + W_\psi(t,x,y,z) + V_\varphi(t,x,y,z), \qquad (7.3.2)$$

where the functions $U_f(t,x,y,z)$, $W_\psi(t,x,y,z)$, $V_\varphi(t,x,y,z)$ have, by definition, the expressions

$$\begin{aligned}
U_f(t,x,y,z) &= \frac{1}{4\pi a^2}\int_{B(x,y,z,at)}\frac{f(\xi,\eta,\zeta,t-r/a)}{r}\mathrm{d}\xi d\eta d\zeta,\\
W_\psi(t,x,y,z) &= \frac{1}{4\pi a^2}\int_{\partial B(x,y,z,at)}\frac{\psi(\xi,\eta,\zeta)}{t}\mathrm{d}\sigma_{at}, \qquad (7.3.3)\\
V_\varphi(t,x,y,z) &= \frac{1}{4\pi a^2}\frac{\partial}{\partial t}\left(\int_{\partial B(x,y,z,at)}\frac{\psi(\xi,\eta,\zeta)}{t}\mathrm{d}\sigma_{at}\right),
\end{aligned}$$

where

$$r = |\xi x| = \sqrt{\sum_{i=1}^{3}(x_i-\xi_i)^2} = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}.$$

Also, in the formulas (7.3.3), $B(x,y,z,at)$ is a ball with the center at the point of coordinates (x,y,z) and the radius $a.t$ and $\partial B(x,y,z,at)$ is the boundary of this ball, that is the sphere with the same center and the same radius.

In the following theorem we show that the function u defined in Eq. (7.3.2), is effective the classical solution of the Cauchy's problem (7.3.1). This is the main result of this paragraph.

Theorem 7.3.1 *If $f \in C^2((0,+\infty))$, $\varphi \in C^3(\mathbb{R}^3)$ and $\psi \in C^3(\mathbb{R}^3)$, then the function u defined in Eq. (7.3.2) is the classical solution of the Cauchy's problem (7.3.1).*

Proof We make the proof in three steps. In the first step we prove that the function W_ψ from Eq. $(7.3.3)_2$ is the solution of the problem

$$\begin{gathered}\frac{\partial^2 W_\psi}{\partial t^2}(t,x,y,z)-a^2\Delta W_\psi(t,x,y,z)=0,\ \forall(t,x,y,z)\in(0,+\infty)\times\mathbb{R}^3,\\ W_\psi(0,x,y,z)=0,\ \forall(x,y,z)\in\mathbb{R}^3,\\ \frac{\partial W_\psi}{\partial t}(0,x,y,z)=\psi(x,y,z),\ \forall(x,y,z)\in\mathbb{R}^3.\end{gathered}\tag{7.3.4}$$

In the second step we prove that the function V_φ from Eq. $(7.3.3)_3$ is the solution of the problem

$$\begin{gathered}\frac{\partial^2 V_\varphi}{\partial t^2}(t,x,y,z)-a^2\Delta V_\varphi(t,x,y,z)=0,\ \forall(t,x,y,z)\in(0,+\infty)\times\mathbb{R}^3,\\ V_\varphi(0,x,y,z)=\varphi(x,y,z),\ \forall(x,y,z)\in\mathbb{R}^3,\\ \frac{\partial V_\varphi}{\partial t}(0,x,y,z)=0,\ \forall(x,y,z)\in\mathbb{R}^3,\end{gathered}\tag{7.3.5}$$

and, in last step, we prove that the function U_f from Eq. $(7.3.3)_1$ is the solution of the problem

$$\begin{gathered}\frac{\partial^2 U_f}{\partial t^2}(t,x,y,z)-a^2\Delta U_f(t,x,y,z)=f(t,x,y,z),\forall(t,x,y,z)\in(0,+\infty)\times\mathbb{R}^3,\\ V_\varphi(0,x,y,z)=\varphi(x,y,z),\ \forall(x,y,z)\in\mathbb{R}^3,\\ \frac{\partial V_\varphi}{\partial t}(0,x,y,z)=0,\ \forall(x,y,z)\in\mathbb{R}^3.\end{gathered}\tag{7.3.6}$$

If we prove the previous three results, then, taking into account Eq. (7.3.2), we will deduce

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}-a^2\Delta u=\frac{\partial^2 U_f}{\partial t^2}-a^2\Delta U_f+\frac{\partial^2 W_\psi}{\partial t^2}-a^2\Delta W_\psi+\\ +\frac{\partial^2 V_\varphi}{\partial t^2}-a^2\Delta V_\varphi=f(t,x,y,z)+0+0=f(t,x,y,z).\end{aligned}$$

Then

$$u(0,x,y,z) = U_f(0,x,y,z) + W_\psi(0,x,y,z) + \\ + V_\varphi(0,x,y,z) = 0 + 0 + \varphi(x,y,z) = \varphi(x,y,z),$$

and, finally,

$$\frac{\partial u}{\partial t}(0,x,y,z) = \frac{\partial U_f}{\partial t}(0,x,y,z) + \frac{\partial W_\psi}{\partial t}(0,x,y,z) + \\ + \frac{\partial V_\varphi}{\partial t}(0,x,y,z) = 0 + \psi(x,y,z) + 0 = \psi(x,y,z),$$

that is, u from Eq. (7.3.2) effectively verifies the problem (7.3.1) and the proof will be concluded.

Step I.

We denote by M the point of the coordinates (x, y, z) and then we can write W_ψ in the form

$$W_\psi(t,x,y,z) = \frac{1}{4\pi a^2 t} \int_{\partial B(M,at)} \psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at},$$

where $\mathrm{d}\sigma_{at}$ is the element of area on the sphere of radius at.

We make the change of variables $(\xi, \eta, \zeta) \to (\alpha, \beta, \gamma)$:

$$\begin{aligned} \xi &= x + \alpha a t, \\ \eta &= y + \beta a t, \\ \zeta &= z + \gamma a t. \end{aligned} \tag{7.3.7}$$

Then

$$\alpha^2 + \beta^2 + \gamma^2 = \frac{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}{a^2 t^2} = 1,$$

that is, the point of the coordinates (α, β, γ) is on the unit sphere $B(M, 1)$. As a consequence, the function W_ψ receives the form

$$W_\psi(t,x,y,z) = \frac{t}{4\pi} \int_{\partial B(M,1)} \psi(x + \alpha a t, y + \beta a t, z + \gamma a t)\mathrm{d}\sigma_1. \tag{7.3.8}$$

Taking into account that the function ψ has been assumed of the class C^2 and is defined on a compact set (the unit sphere), we have

$$\left|W_\psi(t,x,y,z)\right| \le \frac{|t|c_0}{4\pi} \int_{\partial B(M,1)} \mathrm{d}\sigma_1 = tc_0,$$

and, therefore,

$$W_\psi(t, x, y, z) \to 0 \;\; as \;\; t \to 0^+,$$

uniformly with regard to x, y, z, that is, W_ψ satisfies the condition (7.3.4)$_2$.

Now, we derive with respect to t, in Eq. (7.3.8):

$$\frac{\partial W_\psi}{\partial t}(t, x, y, z) = \frac{1}{4\pi}\int_{\partial B(M,1)} \psi(x + \alpha at, y + \beta at, z + \gamma at)\mathrm{d}\sigma_1 +$$
$$+ \frac{at}{4\pi}\int_{\partial B(M,1)} \left[\alpha \frac{\partial\psi(x + \alpha at, y + \beta at, z + \gamma at)}{\partial(x + \alpha at)} + \right.$$
$$\left. +\beta\frac{\partial\psi(x+\alpha at, y+\beta at, z+\gamma at)}{\partial(y+\beta at)} + \gamma\frac{\partial\psi(x+\alpha at, y+\beta at, z+\gamma at)}{\partial(z + \zeta at)}\right]\mathrm{d}\sigma_1. \tag{7.3.9}$$

We denote by I_2 the last integral from Eq. (7.3.9) and observe that its integrant is a derivable function in the direction of the normal. Then

$$|I_2| \le \frac{at}{4\pi}\int_{\partial B(M,1)} \left|\frac{\partial\psi(x + \alpha at, y + \beta at, z + \gamma at)}{\partial\nu}\right| \mathrm{d}\sigma_1 \le$$
$$\le \frac{atc_1}{4\pi}\int_{\partial B(M,1)} \mathrm{d}\sigma_1 = \frac{atc_1}{4\pi}4\pi = atc_1,$$

where c_1 is the supremum of the derivative in the direction of the normal, which exists due to regularity imposed to the function ψ.

Then $I_2 \to 0$, as $t \to 0^+$, uniformly with respect to x, y, z.

For the first integral from the right-hand side of the relation (7.3.9), denoted by I_1, we apply the mean theorem.

Therefore, there exists a point $(\alpha^*, \beta^*, \gamma^*) \in \partial B(M, 1)$ such that

$$I_1 = \frac{1}{4\pi}\int_{\partial B(M,1)} \psi(x + \alpha at, y + \beta at, z + \gamma at)\mathrm{d}\sigma_1 =$$
$$= \frac{1}{4\pi}\psi(x + \alpha^* at, y + \beta^* at, z + \gamma^* at)\int_{\partial B(M,1)} \mathrm{d}\sigma_1 =$$
$$= \psi(x + \alpha^* at, y + \beta^* at, z + \gamma^* at).$$

Then, it is clear that $I_1 \to \psi(x, y, z)$, as $t \to 0^+$, uniformly with respect to x, y, z. In conclusion, if we pass to the limit in Eq. (7.3.9), with $t \to 0^+$, we obtain

$$\lim_{t\to 0^+} \frac{\partial W_\psi}{\partial t}(t, x, y, z) = \psi(x, y, z),$$

that is, W_ψ verifies the initial condition (7.3.4)$_3$.

We outline that Eq. (7.3.9) can be rewritten in the form

$$\frac{\partial W_\psi(t,x,y,z)}{\partial t}=\frac{W_\psi(t,x,y,z)}{t}+\frac{1}{4\pi a t}\int_{\partial B(M,at)}\frac{\partial \psi(\xi,\eta,\zeta)}{\partial \nu}\mathrm{d}\sigma_{at} \tag{7.3.10}$$

after that we return to the variables (ξ, η, ζ).

In the integral from Eq. (7.3.10) we apply the Gauss–Ostrogradski's formula such that Eq. (7.3.10) becomes

$$\frac{\partial W_\psi(t,x,y,z)}{\partial t}=\frac{W_\psi(t,x,y,z)}{t}+\frac{1}{4\pi a t}\int_{\partial B(M,at)}\Delta\psi(\xi,\eta,\zeta)\mathrm{d}\xi d\eta d\zeta. \tag{7.3.11}$$

We denote by $I(t)$ the integral from Eq. (7.3.11) such that (7.3.11) can be written in the form

$$\frac{\partial W_\psi(t,x,y,z)}{\partial t}=\frac{W_\psi(t,x,y,z)}{t}+\frac{1}{4\pi a t}I(t),$$

and, after we derive with respect to t, it follows

$$\frac{\partial^2 W_\psi(t,x,y,z)}{\partial t^2}=\frac{1}{4\pi a t}I'(t). \tag{7.3.12}$$

To compute the derivative in $I(t)$, we use the spherical coordinates:

$$I(t)=\int_{\partial B(M,at)}\Delta\psi(\xi,\eta,\zeta)\mathrm{d}\xi d\eta d\zeta=$$
$$=\int_0^{at}\int_0^{\pi}\int_0^{2\pi}\Delta\psi(r,\theta,\varphi)r\sin\theta dr\mathrm{d}\theta d\varphi.$$

Then

$$I'(t)=a^3t^2\int_0^{\pi}\int_0^{2\pi}\Delta\psi(r,\theta,\varphi)\sin\theta\mathrm{d}\theta d\varphi=$$
$$=a^3t^2\int_{\partial B(M,1)}\Delta\psi\mathrm{d}\sigma_1=a\int_{\partial B(M,at)}\Delta\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at}.$$

Therefore, Eq. (7.3.12) becomes

$$\frac{\partial^2 W_\psi(t,x,y,z)}{\partial t^2}=\frac{1}{4\pi t}\int_{\partial B(M,at)}\Delta\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at}=a^2\Delta W_\psi, \tag{7.3.13}$$

taking into account the Definition $(7.3.3)_2$ for W_ψ and the fact that we can derive under the integral with respect to (ξ, η, ζ), based on the regularity of the function ψ.

The relation (7.3.13) shows that W_ψ satisfies Eq. $(7.3.4)_1$ and the first step is completely proved.

Step II.

Firstly, we can observe that

$$V_\varphi(t,x,y,z) = \frac{\partial W_\varphi(t,x,y,z)}{\partial t}, \tag{7.3.14}$$

taking into account the Definition $(7.3.3)_3$ for V_φ and the Definition $(7.3.3)_2$ written for W_φ (instead of W_ψ).

Then

$$V_\varphi(0,x,y,z) = \frac{\partial W_\varphi(0,x,y,z)}{\partial t} = \varphi(x,y,z),$$

taking into account the first step, that is, V_φ verifies the initial condition $(7.3.5)_2$.

If we derive with respect to t in Eq. (7.3.14), we obtain

$$\frac{\partial V_\varphi(t,x,y,z)}{\partial t} = \frac{\partial^2 W_\varphi(t,x,y,z)}{\partial t^2} = \frac{1}{4\pi a t} I'(t), \tag{7.3.15}$$

after we have used the equality (7.3.12).

Based on the proof from step I, we have

$$I'(t) = a \int_{\partial B(M,at)} \Delta\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at},$$

and then Eq. (7.3.15) becomes

$$\frac{\partial V_\varphi(t,x,y,z)}{\partial t} = \frac{1}{4\pi t}\int_{\partial B(M,at)} \Delta\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at} =$$
$$= \frac{a^2 t}{4\pi}\int_{\partial B(M,1)} \Delta\psi(x+\alpha a t, y+\beta a t, z+\gamma a t)\mathrm{d}\sigma_1.$$

Thus, we will deduce that

$$\frac{\partial V_\varphi(t,x,y,z)}{\partial t} \to 0, \text{ for } t \to 0^+.$$

Since the integral

$$\int_{\partial B(M,1)} \Delta\psi(x+\alpha a t, y+\beta a t, z+\gamma a t)\mathrm{d}\sigma_1$$

is a bounded function, based on the regularity of the function φ.

Therefore V_φ satisfies the initial condition (7.3.5)$_3$.

Taking into account Eq. (7.3.14), it results

$$\frac{\partial^2 V_\varphi}{\partial t^2} - a^2 \Delta V_\varphi = \frac{\partial^2}{\partial t^2}\left(\frac{\partial W_\varphi}{\partial t}\right) - a^2 \Delta \frac{\partial W_\varphi}{\partial t} =$$
$$= \frac{\partial}{\partial t}\left(\frac{\partial^2 W_\varphi}{\partial t^2} - a^2 \Delta W_\varphi\right) = 0,$$

since in the step I we already proved that

$$\frac{\partial^2 W_\varphi}{\partial t^2} - a^2 \Delta W_\varphi = 0.$$

In the conclusion, V_φ satisfies Eq. (7.3.5)$_1$ and the proof steps II is concluded.

Step III.

Firstly, from Eq. (7.3.3)$_1$ we immediately can deduce that

$$\lim_{t\to 0^+} U_f(t,x,y,z) = \frac{1}{4\pi a^2} \lim_{t\to 0^+} \int_{B(M,at)} \frac{f(\xi,\eta,\zeta,t-r/a)}{r} \mathrm{d}\xi d\eta d\zeta = 0,$$

taking into account the regularity of the function f and the fact that, to the limit, the ball $B(M,at)$ reduces to the point (x,y,z).

Therefore U_f satisfies the initial condition (7.3.6)$_2$. We write now U_f in the form

$$U_f(t,x,y,z) = \frac{1}{4\pi a^2} \int_0^t \left\{ \iint_{\partial B(M,\varrho)} \frac{f(\xi,\eta,\zeta,t-\varrho/a)}{\varrho} \mathrm{d}\sigma_\varrho \right\} d\varrho =$$
$$= \frac{1}{4\pi a^2} \int_0^t \left\{ \iint_{\partial B(0,1)} f(x+\alpha\varrho, y+\beta\varrho, z+\gamma\varrho, t-\varrho) \mathrm{d}\sigma_\varrho \right\} \varrho d\varrho. \tag{7.3.16}$$

Then

$$\frac{\partial U_f(t,x,y,z)}{\partial t} = \frac{1}{4\pi a^2} \int_{\partial B(0,1)} f(x+\alpha\varrho, y+\beta\varrho, z+\gamma\varrho, t-\varrho) t \mathrm{d}\sigma_\varrho +$$
$$+ \frac{1}{4\pi a^2} \int_0^t \left\{ \iint_{\partial B(0,1)} f(x+\alpha\varrho, y+\beta\varrho, z+\gamma\varrho, t-\varrho) \mathrm{d}\sigma_\varrho \right\} \varrho d\varrho. \tag{7.3.17}$$

The second integral from Eq. (7.3.17) disappears for $t = 0$. For the first integral we use the mean theorem and then this integral becomes the product between t and a bounded constant and, therefore, tends to zero, for $t \to 0^+$, that is

$$\lim_{t\to 0^+} \frac{\partial U_f(t,x,y,z)}{\partial t} = 0,$$

the limit taking place uniformly with respect to (x, y, z). Therefore U_f satisfies the initial condition (7.3.6)$_3$.

It remains to prove that U_f verifies Eq. (7.3.6)$_1$. To this we introduce the notation

$$U_1(t, \tau, x, y, z) = \\ = \frac{t-\tau}{4\pi a^2} \int_{\partial B(0,1)} f(x+(t-\tau)\xi, y+(t-\tau)\eta, z+(t-\tau)\zeta, t-\varrho) \mathrm{d}\sigma_1. \tag{7.3.18}$$

Then Eq. (7.3.16) becomes

$$U_f(t, x, y, z) = \int_0^t U_1(t, \tau, x, y, z) \mathrm{d}\tau. \tag{7.3.19}$$

Starting from Eq. (7.3.18) we obtain, without difficulty, the relations

$$\begin{gathered} \frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2} - \Delta U_1(t, \tau, x, y, z) = 0, \\ U_1(t, t, x, y, z) = 0, \\ \frac{\partial U_1(t, t, x, y, z)}{\partial t} = f(t, x, y, z). \end{gathered} \tag{7.3.20}$$

Then from Eq. (7.3.19) it results

$$\begin{aligned} \frac{\partial^2 U_f(t, x, y, z)}{\partial t^2} &= \frac{\partial U_1(t, t, x, y, z)}{\partial t} + \int_0^t \frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2} \mathrm{d}\tau = \\ &= f(t, x, y, z) + \int_0^t \frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2} \mathrm{d}\tau. \end{aligned} \tag{7.3.21}$$

On the other hand, we have

$$\begin{aligned} \Delta U_f(t, x, y, z) &= \int_0^t \Delta U_1(t, \tau, x, y, z) \mathrm{d}\tau = \\ &= \int_0^t \frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2} \mathrm{d}\tau, \end{aligned} \tag{7.3.22}$$

where we have taken into account the relation (7.3.20)$_1$.

From Eqs. (7.3.21) and (7.3.22), by substrating, it results

$$\frac{\partial^2 U_f(t, x, y, z)}{\partial t^2} - \Delta U_f(t, x, y, z) = f(t, x, y, z),$$

that is, U_f verifies the nonhomogeneous equation (7.3.6)$_1$ and the proof of the last is over. In the same time, the proof of the theorem is concluded. ■

The formula (7.3.2) which gives the form of the solution for the Cauchy's problem (7.3.1) is called the *Kirchhoff's formula*.

The Kirchhoff's formula is also useful to prove the uniqueness of the solution of the Cauchy's problem. Indeed, if the problem (7.3.1) admits two solutions, $u_1(t, x, y, z)$ and $u_2(t, x, y, z)$, then we denote by $u(t, x, y, z)$ its difference,

$$u(t, x, y, z) = u_1(t, x, y, z) + u_2(t, x, y, z).$$

It is easy to see that $u(t, x, y, z)$ satisfies a Cauchy's problem of the form (7.3.1) where $f \equiv 0$, $\varphi \equiv 0$ and $\psi \equiv 0$. If we write the Kirchhoff's formula for u, obviously, we obtain $u \equiv 0$ from where we will deduce that $u_1 \equiv u_2$.

Finally, the Kirchhoff's formula can be used to prove a result of stability for the solution of the Cauchy's problem (7.3.1), with regard to the right-hand side and initial conditions.

7.4 Problem of the Finite Vibrating Chord

Let us consider the case of the finite chord. The general aim of this paragraph is to study the following initial-boundary values problem, attached to the equation (homogeneous, in first instance) of the finite chord

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \ \forall x \in [0, l], \ \forall t > 0, \\
&u(0, x) = \varphi(x), \ \forall x \in [0, l], \\
&\frac{\partial u}{\partial t}(0, x) = \psi(x), \ \forall x \in [0, l],
\end{aligned} \tag{7.4.1}$$

where the functions $f(t, x)$, $\varphi(x)$ and $\psi(x)$ are given and continuous on their domains of definition. The function $u = u(t, x)$ is the unknown function of the problem and represents the amplitude of the chord at the moment t, at the point x. The positive constant a is prescribed for each type of the material of the chord and the constant l represents the length of the chord.

The mixt initial-boundary value problem is complete if we add the boundary conditions

$$u(t, 0) = g_1(t), \ u(t, l) = g_2(t), \ \forall t > 0,$$

where the functions $g_1(t)$ and $g_2(t)$ are given and describe the behavior of the ends of the chord.

For the sake of simplicity we consider only the case $g_1(t) = g_2(t) = 0$ and we say that the ends of the chord are fixed.

The procedure to solve the above considered problem is based on the Bernoulli-Fourier's method, which is called, also, the "separating the variables" method.

We try to find a solution of the form

$$u(t, x) = X(x)T(t)$$

so that the derivatives become

$$\frac{\partial u}{\partial x} = X'T,\ \frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = X''T,\ \frac{\partial^2 u}{\partial t^2} = XT''.$$

The considered partial differential equation is transformed in an ordinary differential equation

$$XT'' - a^2X''T = 0,$$

which can be restated in the form

$$\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X}.$$

It is easy to see that both sides of this relation are constants, such that we can write

$$X'' - kX = 0, T'' - ka^2T = 0,$$

where the constant k is the common value of the above ratios.

Taking into account the boundary conditions, we obtain

$$u(t, 0) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(t, l) = 0 \Rightarrow X(l)T(t) = 0 \Rightarrow X(l) = 0.$$

In this way, with regard to the function $X(x)$ we have the following problem

$$\begin{aligned} &X'' - kX = 0, \\ &X(0) = 0,\ X(l) = 0. \end{aligned} \tag{7.4.2}$$

The characteristic equation attached to the above differential equation (having constant coefficients) is

$$r^2 - k = 0,$$

such that we must consider three cases.

I. $k = 0$ In this case the equation reduces to $X'' = 0$ so that

$$X(x) = C_1 x + C_2$$

and, taking into account the boundary condition from Eq. (7.4.2), the constants C_1 and C_2 become zero. In conclusion, in this case we obtain the solution $X(x) = 0$ which does not satisfy our problem (7.4.1).

II. $k > 0$ In this case the characteristic equation has two real roots $\pm\sqrt{k}$ and the differential equation has the general solution

$$X(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}.$$

Taking into account the boundary condition from (7.4.1), the constants C_1 and C_2 become zero. Therefore, also in this case we obtain the solution $X(x) = 0$ which does not satisfy our problem (7.4.1).

III. $k < 0$ Denote $k = -\lambda^2$. In this case the characteristic equation has two conjugated complex roots $\pm i\lambda$ and the differential equation has the general solution

$$X(x) = C_1 \cos\lambda + C_2 \sin\lambda.$$

Taking into account the boundary condition from Eq. (7.4.2), in order to determine the constants C_1 and C_2, we obtain the relations $C_1 = 0$ and $C_2 \sin\lambda l = 0$. Then

$$\sin\lambda l = 0 \Rightarrow \lambda = n\pi,\ \ n = 0, 1, 2, ...$$

So, we obtain an infinite number of values for the parameter λ, called *proper values* for the problem of the finite vibrating chord

$$\lambda_n = \frac{n\pi}{l},\ \ n = 0, 1, 2, ...$$

Corresponding, from the general form of the solution, we find an infinite number of functions $X(x)$, called *proper functions*

$$X_n = C_n \sin\frac{n\pi}{l}x,\ \ C_n = \text{constants},\ \ n = 0, 1, 2, ...$$

Taking into account the value of the parameter $k = -\lambda^2$, for the determination of the function $T(t)$ we have the equation

$$T'' - \left(\frac{n\pi a}{l}\right) T = 0,$$

which has the solutions

$$T_n = D_n \cos \frac{n\pi a}{l} t + E_n \sin \frac{n\pi a}{l} t,$$

where $D_n,\ E_n$ =constants, $n = 0, 1, 2, ...$

In conclusion, for our initial mixt problem, there exists an infinite number of particular solutions

$$u_n(t,x) = X_n(x)T_n(t) = \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t\right) \sin \frac{n\pi}{l} x,\ n = 0, 1, 2, ..$$

where we used the notation

$$A_n = C_n D_n,\ B_n = C_n E_n.$$

Since our problem is a linear one, its general solution will be a linear combination of the particular solutions, that is

$$u(t,x) = \sum_{n=0}^{\infty} \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t\right) \sin \frac{n\pi}{l} x.$$

In order to determine the coefficients A_n and $B_n$we can use the initial conditions of the mixt problem. Firstly, we have

$$\varphi(x) = u(0,x) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{l} x.$$

This is the Fourier's series of the function $\varphi(x)$ and then A_n are the Fourier's coefficients of the function $\varphi(x)$. Using the known formula for these coefficients, we obtain

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x \mathrm{d}x.$$

In order to use the other initial condition, we have

$$\frac{\partial u}{\partial t}(t,x) = \sum_{n=0}^{\infty} \left(-A_n \frac{n\pi a}{l} \sin \frac{n\pi a}{l} t + \frac{n\pi a}{l} B_n \cos \frac{n\pi a}{l} t\right) \sin \frac{n\pi}{l} x.$$

Then, the second initial condition leads to

$$\psi(x) = \frac{\partial u}{\partial t}(0,x) = \sum_{n=0}^{\infty} \frac{n\pi a}{l} B_n \sin \frac{n\pi}{l} x.$$

This is the Fourier's series of the function $\psi(x)$ and then

$$\frac{na\pi}{l}B_n$$

are the Fourier's coefficients of the function $\psi(x)$. Using the known formula for these coefficients, we obtain

$$\frac{na\pi}{l}B_n = \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi}{l}x\mathrm{d}x,$$

such that

$$B_n = \frac{2}{na\pi}\int_0^l \psi(x)\sin\frac{n\pi}{l}x\mathrm{d}x.$$

In conclusion, the solution of the homogeneous finite problem of the vibrating chord is

$$u(t,x) = \sum_{n=0}^{\infty}\left(A_n\cos\frac{n\pi a}{l}t + B_n\sin\frac{n\pi a}{l}t\right)\sin\frac{n\pi}{l}x,$$

where the coefficients A_n and B_n have the expressions

$$A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l}x\mathrm{d}x,$$

$$B_n = \frac{2}{na\pi}\int_0^l \psi(x)\sin\frac{n\pi}{l}x\mathrm{d}x.$$

At the end of this paragraph, we consider the nonhomogeneous finite problem of the vibrating chord, that is

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(t,x),\ \forall x \in [0,l],\ \forall t > 0,\\
&u(0,x) = \varphi(x),\ \forall x \in [0,l],\\
&\frac{\partial u}{\partial t}(0,x) = \psi(x),\ \forall x \in [0,l],\\
&u(0,t) = u(l,t) = 0.
\end{aligned}\tag{7.4.3}$$

In order to solve this problem, we decompose it in two problems: one having the differential equation in homogeneous form and nonhomogeneous initial conditions and, the second having nonhomogeneous differential equation and homogeneous

initial conditions:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \ \forall x \in [0, l], \ \forall t > 0, \\
&u(0, x) = \varphi(x), \ \forall x \in [0, l], \\
&\frac{\partial u}{\partial t}(0, x) = \psi(x), \ \forall x \in [0, l], \\
&u(t, 0) = u(t, l) = 0.
\end{aligned} \tag{7.4.4}$$

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x), \ \forall x \in [0, l], \ \forall t > 0, \\
&u(0, x) = 0, \ \forall x \in [0, l], \\
&\frac{\partial u}{\partial t}(0, x) = 0, \ \forall x \in [0, l], \\
&u(t, 0) = u(t, l) = 0.
\end{aligned} \tag{7.4.5}$$

Of course, to solve the problem (7.4.4) we use the above exposed procedure. Let us solve the problem (7.4.5). We will find a particular solution of the form

$$u_p(t, x) = \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi}{l} x, \tag{7.4.6}$$

that is, the unknown coefficients $C_n(t)$ depend only of t.

It is easy to see that

$$u_p(t, 0) = u_p(t, l) = 0.$$

Also, derivating in Eq. (7.4.6) with respect to t, we have

$$\frac{\partial u_p}{\partial t} = \sum_{n=1}^{\infty} C_n'(t) \sin \frac{n\pi}{l} x,$$

and

$$\frac{\partial^2 u_p}{\partial t^2} = \sum_{n=1}^{\infty} C_n''(t) \sin \frac{n\pi}{l} x.$$

Now, derivating in Eq. (7.4.6) with respect to x, we obtain

$$\frac{\partial u_p}{\partial x} = \sum_{n=1}^{\infty} C_n(t) \frac{n\pi}{l} \cos \frac{n\pi}{l} x,$$

and

$$\frac{\partial^2 u_p}{\partial x^2} = -\sum_{n=1}^{\infty} C_n(t) \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi}{l}x.$$

Then, the equation from the problem (7.4.5) reduces to

$$\sum_{n=1}^{\infty} \left[C_n''(t) + \left(\frac{na\pi}{l}\right)^2 C_n(t) \right] \sin\frac{n\pi}{l}x = f(t,x).$$

This relation can be considered as the Fourier's series of the function $f(t,x)$ and then the above square brackets are the Fourier's coefficients of the function $f(t,x)$

$$C_n''(t) + \left(\frac{na\pi}{l}\right)^2 C_n(t) = \frac{2}{l}\int_0^l f(t,x)\sin\frac{n\pi}{l}x = h_n(t),$$

where the last equality being a notation.

Since $u_p(0,x) = 0$ we deduce

$$\sum_{n=1}^{\infty} C_n(0)\sin\frac{n\pi}{l}x = 0 \Rightarrow C_n(0) = 0.$$

Also, since

$$\frac{\partial u_p}{\partial t}(0,x) = 0,$$

we deduce

$$\sum_{n=1}^{\infty} C_n'(0)\sin\frac{n\pi}{l}x = 0 \Rightarrow C_n'(0) = 0.$$

Therefore, to find the coefficients $C_n(t)$ we must solve the simple Cauchy's problem, attached to an ordinary differential equation

$$\begin{aligned} &C_n''(t) + \left(\frac{na\pi}{l}\right)^2 C_n(t) = h_n(t), \\ &C_n(0) = 0, \\ &C_n'(0) = 0. \end{aligned}$$

In conclusion, the solution of the problem (7.4.5) is the function (7.4.6) where the coefficients $C_n(t)$ satisfy the above Cauchy's problem.

It is a simple matter to show that the solution of the mixt initial-boundary values problem of the nonhomogeneous finite vibrating chord (7.4.3) is the sum of the solution of problem (7.4.4) with the solution of the problem (7.4.5).

Chapter 8
Parabolical Equations

8.1 The Finite Problem of Heat

The main exponent of the parabolical equations is the equation of heat conduction in a body.

The general aim of this paragraph is to study the following initial-boundary values problem, attached to the equation (homogeneous, in first instance) of the heat conduction in a rod. This is a bar with its cross section small in comparison with the length.

$$\begin{aligned} &\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \ \forall x \in [0, l], \ \forall t > 0, \\ &u(0, x) = \varphi(x), \ \forall x \in [0, l], \end{aligned} \tag{8.1.1}$$

where the functions $f(t, x)$, $\varphi(x)$ and $\psi(x)$ are given and continuous on their domains of definition. The function $u = u(t, x)$ is the unknown function of the problem and represents the temperature in the rod at the moment t, at the point x. The positive constant a is prescribed for each type of the material of the rod and the constant l represents the length of the rod.

The mixt initial-boundary value problem is complete if we add the boundary conditions

$$u(t, 0) = g_1(t), \ u(t, l) = g_2(t), \ \forall t > 0,$$

where the functions $g_1(t)$ and $g_2(t)$ are given and describe the behavior of the ends of the rod.

For the sake of simplicity we consider only the case $g_1(t) = g_2(t) = 0$ and we say that the ends of the rod are free of temperature.

The procedure to solve the above considered problem is based on the Bernoulli–Fourier's method, which, also, is called the "separating of the variables" method.

M. Marin and A. Öchsner, *Complements of Higher Mathematics*,
https://doi.org/10.1007/978-3-319-74684-5_8

We try to find a solution of the form

$$u(t, x) = X(x)T(t)$$

so that the derivatives become

$$\frac{\partial u}{\partial x} = X'T, \ \frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = X''T.$$

The considered partial differential equation is transformed in an ordinary differential equation

$$XT' - a^2X''T = 0,$$

which can be restated in the form

$$\frac{1}{a^2}\frac{T'}{T} = \frac{X''}{X}.$$

It is easy to see that both sides of this relation are constants, such that we can write

$$X'' - kX = 0, T' - ka^2T = 0,$$

where the constant k is the common value of the above ratios.

Taking into account the boundary conditions, we obtain

$$u(t, 0) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(t, l) = 0 \Rightarrow X(l)T(t) = 0 \Rightarrow X(l) = 0.$$

In this way, with regard to the function $X(x)$ we have the following problem

$$\begin{aligned} &X'' - kX = 0,\\ &X(0) = 0, \ \ X(l) = 0. \end{aligned} \tag{8.1.2}$$

The characteristic equation attached to the above differential equation (having constant coefficients) is

$$r^2 - k = 0,$$

such that we must consider three cases.

I. $k = 0$ In this case the equation reduces to $X'' = 0$ so that

$$X(x) = C_1 x + C_2$$

and, taking into account the boundary condition from Eq. (8.1.2), the constants C_1 and C_2 become zero. In conclusion, in this case we obtain the solution $X(x) = 0$ which does not satisfy our problem (8.1.2).

II. $k > 0$ In this case the characteristic equation has two real roots $\pm\sqrt{k}$ and the differential equation has the general solution

$$X(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}.$$

Taking into account the boundary condition from Eq. (8.1.2), the constants C_1 and C_2 become zero. Therefore, also in this case we obtain the solution $X(x) = 0$ which does not satisfy our problem (8.1.2).

III. $k < 0$ Denote $k = -\lambda^2$. In this case the characteristic equation has two conjugated complex roots $\pm i\lambda$ and the differential equation has the general solution

$$X(x) = C_1 \cos\lambda + C_2 \sin\lambda.$$

Taking into account the boundary condition from Eq. (8.1.2), in order to determine the constants C_1 and C_2, we obtain the relations $C_1 = 0$ and $C_2 \sin\lambda l = 0$. Then

$$\sin\lambda l = 0 \Rightarrow \lambda = n\pi,\ \ n = 0, 1, 2, ...$$

So, we obtain an infinite number of values for the parameter λ, called the *proper values* for the finite problem of the heat conduction:

$$\lambda_n = \frac{n\pi}{l},\ \ n = 0, 1, 2, ...$$

Corresponding, from the general form of the solution, we find an infinite number of functions $X(x)$, called the *proper functions*

$$X_n = C_n \sin\frac{n\pi}{l}x,\ \ C_n = \text{constants},\ \ n = 0, 1, 2, ...$$

Taking into account the value of the parameter $k = -\lambda^2$, for the determination of the function $T(t)$ we have the equation

$$T' - \left(\frac{n\pi a}{l}\right) T = 0,$$

from where it results

$$T_n = D_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}, \ D_n = \text{constants}, \ n = 0, 1, 2, ...$$

In conclusion, for our initial mixt problem, there exists an infinite number of particular solutions

$$u_n(t, x) = X_n(x)T_n(t) = A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi}{l}x, \ n = 0, 1, 2, ..$$

where we used the notation

$$A_n = C_n D_n.$$

Since our problem is linear, its general solution will be a linear combination of the particular solutions, that is

$$u(t, x) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi}{l}x.$$

In order to determine the coefficients A_n we can use the initial condition of the mixt problem. So, we have

$$\varphi(x) = u(0, x) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{l}x.$$

This is the Fourier's series of the function $\varphi(x)$ and then A_n are the Fourier's coefficients of the function $\varphi(x)$. Using the known formula for these coefficients, we obtain

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l}x\,\mathrm{d}x.$$

In conclusion, the solution of the homogeneous finite problem of the heat conduction is

$$u(t, x) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi}{l}x, \quad A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l}x\,\mathrm{d}x.$$

At the end of this paragraph, we consider the nonhomogeneous finite problem of the heat conduction, that is

$$\begin{aligned} &\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x), \ \forall x \in [0, l], \ \forall t > 0, \\ &u(0, x) = \varphi(x), \ \forall x \in [0, l], \\ &u(0, t) = u(l, t) = 0. \end{aligned} \tag{8.1.3}$$

In order to solve this problem, we decompose it in two problems: one having the differential equation in its homogeneous form and nonhomogeneous initial conditions and, the second having a nonhomogeneous differential equation and homogeneous initial conditions:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \ \forall x \in [0, l], \ \forall t > 0, \\
&u(0, x) = \varphi(x), \ \forall x \in [0, l], \\
&u(t, 0) = u(t, l) = 0.
\end{aligned} \tag{8.1.4}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x), \ \forall x \in [0, l], \ \forall t > 0, \\
&u(0, x) = 0, \ \forall x \in [0, l], \\
&u(t, 0) = u(t, l) = 0.
\end{aligned} \tag{8.1.5}$$

Of course, in order to solve the problem (8.1.4) we use the above exposed procedure. Let us solve the problem (8.1.5). We will find a particular solution of the form

$$u_p(t, x) = \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi}{l} x, \tag{8.1.6}$$

that is, the unknown coefficients $C_n(t)$ depend only on t.
It is easy to see that

$$u_p(t, 0) = u_p(t, l) = 0.$$

Also, derivating in Eq. (8.1.6) with respect to t, we have

$$\frac{\partial u_p}{\partial t} = \sum_{n=1}^{\infty} C_n'(t) \sin \frac{n\pi}{l} x.$$

Now, derivating in Eq. (8.1.6) with respect to x, we obtain

$$\frac{\partial u_p}{\partial x} = \sum_{n=1}^{\infty} C_n(t) \frac{n\pi}{l} \cos \frac{n\pi}{l} x,$$

and

$$\frac{\partial^2 u_p}{\partial x^2} = -\sum_{n=1}^{\infty} C_n(t) \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi}{l} x.$$

Then, the equation from (8.1.5) reduces to

$$\sum_{n=1}^{\infty}\left[C_n'(t)+\left(\frac{na\pi}{l}\right)^2 C_n(t)\right]\sin\frac{n\pi}{l}x = f(t,x). \tag{8.1.7}$$

We multiply both sides of this equality by the functions

$$\sin\frac{m\pi}{l}x$$

and use the fact that these functions are orthogonal:

$$\int_0^l \sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\mathrm{d}x = \frac{1}{2}\int_0^l\left[\cos(n-m)\frac{n\pi}{l}x-\cos(n+m)\frac{n\pi}{l}x\right]\mathrm{d}x.$$

In order to evaluate the last integral, we must consider two cases. Firstly, if $n\neq m$ we obtain

$$\frac{1}{2}\int_0^l\left[\cos(n-m)\frac{n\pi}{l}x-\cos(n+m)\frac{n\pi}{l}x\right]\mathrm{d}x =$$

$$=\frac{1}{2}\left[\frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi}{l}x\Big|_0^l-\frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi}{l}x\Big|_0^l\right]=0.$$

If $n=m$ we have

$$\int_0^l \sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\mathrm{d}x = \int_0^l\sin^2\frac{n\pi}{l}x\mathrm{d}x = \frac{1}{2}\int_0^l\left(1-\cos\frac{2n\pi}{l}x\right)\mathrm{d}x =$$

$$=\frac{1}{2}\left[x\Big|_0^l-\frac{l}{2n\pi}\sin\frac{2n\pi}{l}x\Big|_0^l\right]=\frac{l}{2}-\frac{l}{4n\pi}\sin\frac{2n\pi}{l}x\Big|_0^l=\frac{l}{2}.$$

In conclusion, the scalar product takes the following value

$$\left(\sin\frac{n\pi}{l}x,\ \sin\frac{m\pi}{l}x\right)=\int_0^l\sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\mathrm{d}x=\begin{cases}0, & n\neq m\\ l/2, & n=m.\end{cases}$$

After these calculations, from Eq. (8.1.7) we obtain

$$\int_0^l\sum_{n=1}^{\infty}\left[C_n'(t)+\left(\frac{na\pi}{l}\right)^2 C_n(t)\right]\sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\mathrm{d}x=\int_0^l f(t,x)\sin\frac{m\pi}{l}x\mathrm{d}x \Rightarrow$$

$$\Rightarrow\sum_{n=1}^{\infty}\left[C_n'(t)+\left(\frac{na\pi}{l}\right)^2 C_n(t)\right]\int_0^l\sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\mathrm{d}x=\int_0^l f(t,x)\sin\frac{m\pi}{l}x\mathrm{d}x.$$

Using the above result with regard to integral in the left-hand side, we deduce

$$\left[C_n'(t) + \left(\frac{na\pi}{l}\right)^2 C_n(t)\right]\frac{l}{2} = \int_0^l f(t,x)\sin\frac{n\pi}{l}x\mathrm{d}x,$$

which can be restated in the form

$$C_n'(t) + \left(\frac{na\pi}{l}\right)^2 C_n(t) = \frac{2}{l}\int_0^l f(t,x)\sin\frac{n\pi}{l}x\mathrm{d}x.$$

Using the notation

$$H_n(t) = \frac{2}{l}\int_0^l f(t,x)\sin\frac{n\pi}{l}x\mathrm{d}x,$$

the above equation can be written in the form

$$C_n'(t) + \left(\frac{na\pi}{l}\right)^2 C_n(t) = H_n(t),$$

which is a linear and nonhomogeneous differential equation of first order. It is well known that a linear and homogeneous differential equation

$$y' + P(x)y = Q(x)$$

has the solution

$$y(x) = e^{-\int_0^x P(t)\mathrm{d}t}\left[C + \int_0^x Q(t)e^{\int_0^t P(t)\mathrm{d}t}\mathrm{d}t\right],\ C = \text{constant}.$$

In our case, we have (taking $C = 0$):

$$C_n(y) = \int_0^t e^{-(na\pi/l)^2 t}H_n(s)e^{(na\pi/l)^2 s}\mathrm{d}s = \int_0^t H_n(s)e^{-(na\pi/l)^2(t-s)}\mathrm{d}s.$$

Then, the particular solution of the nonhomogeneous equation is

$$u_p(t,x) = \sum_0^\infty C_n(t)\sin\frac{n\pi}{l}x =$$

$$= \sum_0^\infty\left[\int_0^t H_n(s)e^{-(na\pi/l)^2(t-s)}\mathrm{d}s\right]\sin\frac{n\pi}{l}x\mathrm{d}s =$$

$$= \int_0^l\left[\sum_0^\infty H_n(s)e^{-(na\pi/l)^2(t-s)}\sin\frac{n\pi}{l}x\right]\mathrm{d}s.$$

We now remember that the general solution of the homogeneous equation is

$$u_0(t,x)=\sum_0^\infty A_n(t)e^{-(na\pi/l)^2t}\sin\frac{n\pi}{l}x,\ A_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\mathrm{d}x,$$

and this can be restated in the form

$$u_0(t,x)=\sum_0^\infty\frac{2}{l}\int_0^l\varphi(s)\sin\frac{n\pi}{l}se^{-(na\pi/l)^2t}\mathrm{d}s.$$

Then, taking into account that the general solution of the nonhomogeneous equation is

$$u(t,x)=u_0(t,x)+u_p(t,x),$$

we can write

$$u(t,x)=\sum_0^\infty\left[A_n+\int_0^t H_n(s)e^{(na\pi/l)^2s}\mathrm{d}s\right]e^{-(na\pi/l)^2t}\sin\frac{n\pi}{l}x,$$

where

$$A_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\mathrm{d}x,\ H_n=\frac{2}{l}\int_0^l f(t,x)\sin\frac{n\pi}{l}x\mathrm{d}x.$$

In the following we will use an operational method to solve the problem of the heat conduction in a semi-infinite one-dimensional rod. In fact we use the Laplace's transform to solve this problem.

Let us consider a semi-infite one-dimensional rod free of temperature at the initial moment. The variation of temperature in this rod can be computed by the following mathematical model

$$\begin{aligned}&\frac{\partial^2u}{\partial x^2}=\frac{1}{a^2}\frac{\partial u}{\partial t},\ (t,x)\in(0,\infty)\times(0,\infty)\\&u(0,x)=0,\ x>0\\&u(t,0)=u_0,\ t>0.\end{aligned}$$

Here $u=u(t,x)$ is the unknown function of the problem and represents the temperature in the rod at the moment t in the point x of the rod. Also, u_0 is a given constant and represents the value of the temperature at the end of the rod in any moment. Denote

$$U(p,x)=\mathcal{L}_{u(t,x)},$$

that is, the Laplace's transform of the function $u(t,x)$. As we know this integral's transform is

$$\mathcal{L}_{u(t,x)} = \int_0^\infty u(t,x)e^{-pt}\mathrm{d}t.$$

It is easy to prove the following results

$$\mathcal{L}\left(\frac{\partial u}{\partial t}\right) = p\mathcal{L}_{u(t,x)} - u(0,x) = pU(p,x),$$

$$\mathcal{L}\left(\frac{\partial u}{\partial x}\right) = \int_0^\infty \frac{\partial u}{\partial x}e^{-pt}\mathrm{d}t =$$

$$= \frac{\partial}{\partial x}\int_0^\infty u(t,x)e^{-pt}\mathrm{d}t = \frac{\partial U}{\partial x}.$$

Of course, we can write

$$\frac{\partial U}{\partial x} = \frac{dU}{dx},$$

because p is considered as a parameter. Therefore we have

$$\mathcal{L}\left(\frac{\partial u}{\partial x}\right) = \frac{dU}{dx}.$$

Similarly,

$$\mathcal{L}\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_0^{infty} \frac{\partial^2 u}{\partial x^2}e^{-pt}\mathrm{d}t =$$

$$= \frac{\partial^2}{\partial x^2}\int_0^{infty} u(t,x)e^{-pt}\mathrm{d}t = \frac{\partial^2 U}{\partial x^2} = \frac{d^2 U}{dx^2}.$$

After these calculations the heat conduction equation becomes

$$\frac{d^2 U}{dx^2} = \frac{1}{a^2}U(p,x),$$

which is an ordinary differential equation having constant coefficients. Its characteristic equation is

$$r^2 = \frac{p}{a^2} \Rightarrow r = \pm\frac{\sqrt{p}}{a}$$

such that the general solution is

$$U(p,x) = C_1 e^{-\frac{\sqrt{p}}{a}x} + C_2 e^{\frac{\sqrt{p}}{a}x}.$$

This solution must by finite when $x \to \infty$ and then we obtain $C_2 = 0$ and the solution becomes

$$U(p, x) = C_1 e^{-\frac{\sqrt{p}}{a}x}.$$

Now, we can use the condition on the end $x = 0$

$$\mathcal{L}_{u(t,0)} = \mathcal{L}_{u_0} = \frac{u_0}{p},$$

that is

$$C_1 e^{-\frac{\sqrt{p}}{a}0} = \frac{u_0}{p} \Rightarrow C_1 = \frac{u_0}{p}.$$

Finally, the solution of the equation is

$$U(p, x) = \frac{u_0}{p} e^{-\frac{\sqrt{p}}{a}x}.$$

Then, the solution of the heat conduction equation is

$$u(t, x) = \mathcal{L}^{-1}\left[\frac{u_0}{p} e^{-\frac{\sqrt{p}}{a}x}\right].$$

In order to compute the above inverse Laplace's transform, we remember that

$$\mathcal{L}\left(t^{\alpha}\right) = \frac{\Gamma(\alpha + 1)}{p^{\alpha+1}}.$$

Indeed, starting from the definition, we have

$$\mathcal{L}\left(t^{\alpha}\right) = \int_0^{\infty} t^{\alpha} e^{-pt} \mathrm{d}t,$$

where we make the change of variable $pt = \tau$ such that

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{p} \Rightarrow \mathcal{L}\left(t^{\alpha}\right) = \frac{1}{p^{\alpha+1}} \int_0^{\infty} \tau^{\alpha} e^{-\tau} \mathrm{d}\tau = \frac{\Gamma(\alpha + 1)}{p^{\alpha+1}}.$$

In the particular case $\alpha = -1/2$ we obtain

$$\mathcal{L}\left(t^{-1/2}\right) = \frac{\Gamma(1/2)}{\sqrt{p}} = \frac{\sqrt{\pi}}{\sqrt{p}},$$

such that we deduce

$$\frac{1}{\sqrt{p}} = \mathcal{L}\left(\frac{1}{\sqrt{\pi t}}\right).$$

Now, we use the well known series

$$e^u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + ... + \frac{1}{n!}u^n + ...$$

where $u = -\frac{\sqrt{p}}{a}x$. The previous series becomes

$$e^{-\frac{\sqrt{p}}{a}x} = 1 - \frac{1}{1!}\frac{\sqrt{p}}{a}x + \frac{1}{2!}\frac{p}{a^2}x^2 - \frac{1}{3!}\frac{\sqrt{p}}{a}\frac{p}{a^2}x^3 + ...$$

Then

$$\frac{1}{p}e^{-\frac{\sqrt{p}}{a}x} = \frac{1}{p} - \frac{x}{a}\frac{1}{\sqrt{p}} + \frac{x^2}{2!a^2} - \frac{x^3}{3!a^3}\sqrt{p} + ...$$

So, we can write

$$\frac{u_0}{p}e^{-\frac{\sqrt{p}}{a}x} = u_0\left[\mathcal{L}(1) - \frac{x}{a}\mathcal{L}\left(\frac{1}{\sqrt{\pi t}}\right) + ...\right] =$$

$$= u_0\mathcal{L}\left[1 - \frac{x}{a}\frac{1}{\sqrt{\pi t}} + ...\right] = \mathcal{L}\left[u_0\left(1 - \frac{x}{a}\frac{1}{\sqrt{\pi t}} + ...\right)\right].$$

So, we deduce that the solution (above written as an inverse of the Laplace's transform) becomes

$$u(t,x) = u_0\left(1 - \frac{x}{a}\frac{1}{\sqrt{\pi t}} + ...\right).$$

We can write this result in the form

$$u(t,x) = u_0\left[1 - \text{erf}\left(\frac{x}{2a\sqrt{t}}\right) + ...\right],$$

where erf is the function of errors, more utilised in the numerical analysis, given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\tau}\mathrm{d}\tau.$$

8.2 Initial-Boundary Value Problems

Let Ω be a bounded domain from $\mathbb{R}^n$ having the boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. For the temporally constant $T > 0$, arbitrarily fixed, consider the temporally interval $\mathcal{T}_T$ given by

$$\mathcal{T}_T = \{t\ : 0 < t \leq T\}, \overline{\mathcal{T}_T} = \{t\ : 0 \leq t \leq T\}.$$

Then, the equation of heat conduction (shortly, the heat equation) is

$$u_t(t,x) - a^2 \Delta u(t,x) = f(t,x), \ \forall (t,x) \in \mathcal{T}_T \times \Omega. \tag{8.2.1}$$

Here we have used the notation $u_t = \partial u / \partial t$, a is a positive given constant, and Δ is the Laplace's operator.

Usually, to the Eq. (8.2.1) we add the initial condition in the form:

$$u(0,x) = \varphi(x), \ \forall x \in \overline{\Omega}. \tag{8.2.2}$$

The boundary conditions have the following form:

- Dirichlet's condition

$$u(t,y) = \alpha(t,y), \ \forall (t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega; \tag{8.2.3}$$

- Neumann's condition

$$\frac{\partial u}{\partial \nu}(t,y) = \beta(t,y), \ \forall (t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega; \tag{8.2.4}$$

- Mixt condition

$$\lambda_1 \frac{\partial u}{\partial \nu}(t,y) + \lambda_2 u(t,y) = \gamma(t,y), \ \forall (t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega; \tag{8.2.5}$$

If we consider, for instance, the problem (8.2.1), (8.2.2) and (8.2.3), then we have the following physical significations:

- $u(t,x)$, which is the unknown function of the problem, represents the temperature in the body Ω, at any moment t;
- $\varphi(x)$ represents the (known) temperature at the initial moment in all points of the body (the points of boundary included);
- $\alpha(t,y)$ represents the (known) temperature at any moment on the surface $\partial\Omega$ which encloses the body.

Therefore, the problem (8.2.1), (8.2.2) and (8.2.3) consists of the determination of the temperature in all points of the body Ω, at any moment, knowing the temperature in the body at the initial moment and also, knowing at any moment the temperature on the surface of the body, $\partial\Omega$.

In all that follows we will study, especially, the problem (8.2.1), (8.2.2) and (8.2.3). In view of the characterization of this problem we will consider, for the moment, the following standard hypotheses:

(i) the function $f : \mathcal{T}_T \times \partial\Omega \to \mathbb{R}$ is given and $f \in C(\mathcal{T}_T \times \partial\Omega)$;
(ii) the function $\varphi : \overline{\Omega} \to \mathbb{R}$ is given and $\varphi \in C(\overline{\Omega})$;
(iii) the function $\alpha : \overline{\mathcal{T}_T} \times \partial\Omega \to \mathbb{R}$ is given and $\alpha \in C(\overline{\mathcal{T}_T} \times \partial\Omega)$.

We call *the classical solution* of the problem (8.2.1), (8.2.2) and (8.2.3) a function $u = u(t, x)$, $u : \overline{\mathcal{T}_T} \times \overline{\Omega} \to \mathbb{R}$, having the properties:

- $u \in C(\overline{\mathcal{T}_T} \times \overline{\Omega})$;
- $u_t, u_{x_i x_i} \in C(\mathcal{T}_T \times \Omega)$;
- u satisfies the Eq. (8.2.1), the initial condition (8.2.2) and the boundary condition (8.2.3).

In the formulation of the problem (8.2.1), (8.2.2) and (8.2.3), the boundary and initial values are given on the set $\overline{\mathcal{T}_T} \times \partial\Omega$ or on the set $\{0\} \times \overline{\Omega}$.
We define the set Γ by

$$\Gamma = \left\{(t, x) \ : \ (t, x) \in \left(\overline{\mathcal{T}_T} \times \partial\Omega\right) \cup \left(\{0\} \times \overline{\Omega}\right)\right\}, \tag{8.2.6}$$

and we call it *the parabolical boundary*, which is different of the topological boundary. In fact, to obtain the parabolical boundary we remove from the topological boundary "the lid" for $t = T$.

In the following theorem we prove a result with regard to the extreme values, in the case of the homogeneous parabolical equations.

$$u_t(t, x) - \Delta u(t, x) = 0, \ \forall (t, x) \in \mathcal{T}_T \times \Omega. \tag{8.2.7}$$

Theorem 8.2.1 *Consider Ω and $\mathcal{T}_T$, defined as above, and consider the function u such that $u \in C(\overline{\mathcal{T}}_T \times \overline{\Omega})$, $u_t, u_{x_i x_i} \in C(\mathcal{T}_T \times \Omega)$. If u satisfies the homogeneous equation (8.1.7), then the extreme values*

$$\sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} u(t, x), \qquad \inf_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} u(t, x)$$

are taken, necessarily, on Γ.

Proof If we make the proof for the supremum, the result for the infimum immediately follows by substituting u to $-u$.

Firstly, we must outline that in the conditions of the theorem, u takes effective its extreme values, according to the classical Weierstrass's theorem.

We suppose, ad absurdum, that u takes its supremum in the parabolical inside, not on the boundary Γ. This means that we can suppose that there exists a point $(t_0, x^0) \in \overline{\mathcal{T}}_T \times \overline{\Omega} \setminus \Gamma$ such that

$$M = \sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} u(t, x) = u(t_0, x^0).$$

We denote by m the supremum value of the function u taken on Γ:

$$m = \sup_{(t,x)\in\Gamma} u(t, x).$$

According to above supposition, we have

$$M > m. \tag{8.2.8}$$

In the following we will prove that Eq. (8.2.8) leads to a contradiction. We define the function $v(t, x)$ by

$$v(t, x) = u(t, x) + \frac{M - m}{2d^2} \sum_{i=1}^{n} \left(x_i - x_i^0\right)^2, \tag{8.2.9}$$

where d is the diameter of the set $\overline{\Omega}$.

Evaluating the function v on Γ, we obtain

$$v(t, x)|_\Gamma \le m + \frac{M - m}{2} = \frac{M + m}{2} < \frac{M + M}{2} = M. \tag{8.2.10}$$

On the other hand,

$$v(t_0, x^0) = u(t_0, x^0) + \frac{M - m}{2d^2} \sum_{i=1}^{n} \left(x_i^0 - x_i^0\right)^2 = M,$$

that is v, which verifies the same conditions of regularity like u, takes its largest value at the point (t_0, x^0) like u. Since on Γ the values of v are strictly less than M, we will deduce that there exists a point (t_1, x^1) in the parabolical inside such that

$$\sup_{(t,x)\in \overline{\mathcal{T}}_T \times \overline{\Omega}} v(t, x) = v(t_1, x^1),$$

while v cannot take its supremum value on Γ. We write the condition of the extremal value for $v(t, x)$ at the point (t_1, x^1):

$$\left.\frac{\partial v(t, x)}{\partial t}\right|_{(t_1, x^1)} \ge 0. \tag{8.2.11}$$

If $t_1 \in (0, T)$ then in Eq. (8.2.11) we have equality whence it follows the Fermat's condition. If $t_1 = T$, then the value to the right-hand of T does not exist and then a extremum point in t_1 means that the function v is positive and increasing at the left-hand of T. On the other hand, the function $v(t_1, x)$, (like only a function of n spatial variable $(x_1, x_2, ..., x_n)$ takes its supremum on $\overline{\Omega}$ at the point $(x_1^1, x_2^1, ..., x_n^1)$ whence it follows the necessary maximum condition

$$\left.\frac{\partial^2 v(t, x)}{\partial x_i^2}\right|_{(t_1, x^1)} \le 0, \ i = 1, 2, ..., n,$$

from where we obtain

$$\Delta v(t_1, x^1) \leq 0. \tag{8.2.12}$$

From Eqs. (8.2.11) and (8.2.12) it follows

$$(-v_t(t, x) + \Delta v(t, x))_{(t_1, x^1)} \leq 0. \tag{8.2.13}$$

Starting from the form (8.2.9) of the function v, we obtain

$$\begin{aligned}(-v_t(t, x) + \Delta v(t, x))_{(t_1, x^1)} &= (-u_t(t, x) + \Delta u(t, x))_{(t_1, x^1)} + \\ &+ \frac{(M - m)n}{d^2} = \frac{(M - m)n}{d^2} > 0,\end{aligned}$$

in which we have taken into account Eq. (8.2.8).

This inequality is the contrary of the inequality (8.2.13), that proves that the assumption (8.2.8) is falls and the theorem has been proved. ■

As a direct consequence of the theorem of the extreme values, we will prove the uniqueness of the solution for an initial boundary values problem.

Theorem 8.2.2 *The problem constituted by the Eq. (8.2.1), the initial condition (8.2.2) and the boundary condition (8.2.3) has at the most one classical solution.*

Proof We suppose that the problem (8.2.1), (8.2.2) and (8.2.3) admits two classical solutions $u_1(t, x)$ and $u_2(t, x)$. Then we have

$$\begin{aligned}\Delta u_i(t, x) - \tfrac{\partial u_i}{\partial t}(t, x) = f(t, x), \ \forall (t, x) \in \mathcal{T}_T \times \Omega,\\ u_i(0, x) = \varphi(x), \ \forall x \in \overline{\Omega},\\ u_i(t, y) = \alpha(t, y), \ \forall (t, y) \in \overline{\mathcal{T}}_T \times \partial\Omega,\end{aligned} \tag{8.2.14}$$

where $i = 1, 2$ and the functions f, φ and α are given and continuous where they are defined.

On the other hand, u_1 and u_2 satisfy the conditions of a classical solution. We define the function $v(t, x)$ by

$$v(t, x) = u_1(t, x) - u_2(t, x), \ \forall (t, x) \in \overline{\mathcal{T}}_T \times \overline{\Omega}.$$

Taking into account the above considerations, we obtain that v satisfies the conditions of regularity of a classical solution and, more, verifies the problem

$$\begin{aligned}\Delta v(t, x) - \tfrac{\partial v}{\partial t}(t, x) = 0, \ \forall (t, x) \in \mathcal{T}_T \times \Omega,\\ v(0, x) = 0, \ \forall x \in \overline{\Omega},\\ v(t, y) = 0, \ \forall (t, y) \in \overline{\mathcal{T}}_T \times \partial\Omega.\end{aligned} \tag{8.2.15}$$

The function v satisfies all the conditions of the Theorem 8.2.1. Therefore, its extreme values

$$\sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x), \qquad \inf_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x)$$

are, necessarily, reached on Γ. According to Eqs. $(8.2.15)_2$ and $(8.2.15)_3$, we deduce that v is null on the parabolical boundary and then

$$\sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x) = \inf_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x) = 0,$$

that is $v(t,x) = 0,\ \forall(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}$ and de aici $u_1(t,x)\equiv u_2(t,x)$. ■

As a further application of the theorem of the extreme values, we will prove now a result of stability with regard to initial conditions and boundary conditions, for the problem (8.2.1), (8.2.2) and (8.2.3).

Theorem 8.2.3 *We suppose that the function $f(t,x)$ is given and continuous on $\mathcal{T}_T\times\Omega$. Let $\varphi_1(t,x)$ and $\varphi_2(t,x)$ be two functions, given and continuous on $\overline{\Omega}$ and the functions $\alpha_1(t,x)$ and $\alpha_2(t,x)$ given and continuous on $\overline{\mathcal{T}_T}\times\partial\Omega$.*

Consider the following problems

$$\Delta u_i(t,x) - \frac{\partial u_i}{\partial t}(t,x) = f(t,x),\ \forall(t,x)\in\mathcal{T}_T\times\Omega,$$
$$u_i(0,x) = \varphi_i(x),\ \forall x\in\overline{\Omega},$$
$$u_i(t,y) = \alpha_i(t,y),\ \forall(t,y)\in\overline{\mathcal{T}}_T\times\partial\Omega,$$

where $i = 1,2$.

If $\forall\varepsilon > 0,\ \exists\delta = \delta(\varepsilon)$ such that

$$|\varphi(x)| = |\varphi_1(x) - \varphi_2(x)| < \delta,$$
$$|\alpha(x)| = |\alpha_1(x) - \alpha_2(x)| < \delta.$$

Then

$$|u(x)| = |u_1(x) - u_2(x)| < \varepsilon.$$

Proof The function $u(t,x)$ defined, as in the enunciation of the theorem, by

$$u(t,x) = u_1(t,x) - u_2(t,x),$$

satisfies the conditions of a classical solution. Also, u satisfies the problem

$$\begin{aligned} \Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) &= f(t,x) - f(t,x) = 0, \\ u(0,x) = u_1(0,x) - u_2(0,x) &= \varphi_1(x) - \varphi_2(x) = \varphi(x), \\ u(t,y) = u_1(t,y) - u_2(t,y) &= \alpha_1(t,y) - \alpha_2(t,y) = \alpha(t,y). \end{aligned} \tag{8.2.16}$$

Since u satisfies the above conditions of regularity and the homogeneous equation (8.2.16)$_1$, we will deduce that there are satisfied the conditions of the theorem of extreme values. Then, the extreme values of the function u are reached on the parabolical boundary Γ. But on Γ the function u reduces to φ or to α and, because φ and α satisfy the conditions $|\varphi| < \delta$, $|\alpha| < \delta$, we obtain the result of the theorem, taking $\delta = \varepsilon$. ∎

A *particular solution* of the problem consists of Eqs. (8.2.1), (8.2.2) and (8.2.3) is the solution obtained by fixing the right-hand term f of the equation, of the initial data φ and of the boundary data α.

The family of all particular solutions obtained by the variation of the functions f, φ and α, in the class of the continuous functions, is *the general solution* of the problem (8.2.1), (8.2.2) and (8.2.3).

Now, we prove that a particular solution of the homogeneous equation of the heat conduction is the function V defined by

$$V(t,\tau,x,\xi) = \frac{1}{\left(2\sqrt{\pi}\right)^n \left(\sqrt{t-\tau}\right)^n} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\xi_i)^2}{4(t-\tau)}\right). \tag{8.2.17}$$

Proposition 8.2.1 *The function $V(t,\tau,x,\xi)$, for $0 \le \tau < t \le T$, is of the class C^∞ and satisfies the equations:*

$$\begin{aligned} \Delta_x V(t,\tau,x,\xi) - \frac{\partial V(t,\tau,x,\xi)}{\partial t} &= 0, \\ \Delta_\xi V(t,\tau,x,\xi) + \frac{\partial V(t,\tau,x,\xi)}{\partial \tau} &= 0. \end{aligned}$$

Proof After some elementary calculations, we obtain

$$\frac{\partial V(t,\tau,x,\xi)}{\partial x_i} = V(t,\tau,x,\xi)\left(-\frac{(x_i-\xi_i)}{2(t-\tau)}\right) = -\frac{\partial V(t,\tau,x,\xi)}{\partial \xi_i}.$$

Therefore

$$\frac{\partial^2 V(t,\tau,x,\xi)}{\partial x_i^2} = V(t,\tau,x,\xi)\left(\frac{(x_i-\xi_i)^2}{4(t-\tau)^2} - \frac{1}{2(t-\tau)}\right) = \frac{\partial^2 V(t,\tau,x,\xi)}{\partial \xi_i^2}.$$

By adding the relations obtained for $i = 1, 2, ..., n$ it follows

$$\Delta_x V(t,\tau,x,\xi) = V(t,\tau,x,\xi)\left(\frac{1}{4(t-\tau)^2}\sum_{i=1}^{n}(x_i-\xi_i)^2 - \frac{n}{2(t-\tau)}\right) =$$
$$= \Delta_\xi V(t,\tau,x,\xi).$$

On the other hand, using the derivative in Eq. (8.2.17) with regard respectively, to t and τ, it follows:

$$\frac{\partial V(t,\tau,x,\xi)}{\partial t} = V(t,\tau,x,\xi)\left(\frac{1}{4(t-\tau)^2}\sum_{i=1}^{n}(x_i-\xi_i)^2 - \frac{n}{2(t-\tau)}\right) =$$
$$= -\frac{\partial V(t,\tau,x,\xi)}{\partial \tau}.$$

So, the results from the enunciation of the proposition are immediately obtained. The fact that the function $V(t,\tau,x,\xi)$ is of the class C^∞ can be argued using the fact that $t \neq \tau$ and, essentially, $V(t,\tau,x,\xi)$ is an exponential function. ∎

Remark. It is easy to verify the fact that if $x \neq \xi$, then the function $V(t,\tau,x,\xi)$ is superior bounded by an exponential function and

$$\lim_{t-\tau\to 0^+} V(t,\tau,x,\xi) = 0.$$

If $x = \xi$, then the exponential function disappears and

$$\lim_{t-\tau\to 0^+} V(t,\tau,x,\xi) = +\infty.$$

Another important property of the function $V(t,\tau,x,\xi)$ will be proved in the following theorem.

Theorem 8.2.4 *The following equalities are true*

$$\int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}x = 1, \quad \int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi = 1.$$

Proof We write the volume integral in extension

$$\int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi =$$
$$= \frac{1}{(2\sqrt{\pi})^n}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}\frac{1}{(\sqrt{t-\tau})^n}\exp\left(-\frac{\sum_{i=1}^{n}(x_i-\xi_i)^2}{4(t-\tau)}\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\ldots\mathrm{d}\xi_n.$$

We make the change of variable $\xi_i - x_i = 2\sqrt{t-\tau}\eta_i$ and by direct calculations, we obtain that the Jacobian of the change has the value:

$$\left|\frac{\mathrm{d}\xi}{D\eta}\right| = 2^n \left(\sqrt{t-\tau}\right)^n.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi &= \frac{1}{\left(\sqrt{\pi}\right)^n}\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{-\sum\limits_{i=1}^{n}\eta_i^2} d\eta_1 d\eta_2 \dots d\eta_n = \\ &= \frac{1}{\left(\sqrt{\pi}\right)^n}\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{-\eta_1^2} e^{-\eta_2^2} \dots e^{-\eta_n^2} d\eta_1 d\eta_2 \dots d\eta_n = \\ &= \frac{1}{\left(\sqrt{\pi}\right)^n}\left(\int_{-\infty}^{+\infty} e^{-s^2}\mathrm{d}s\right)^n = \frac{1}{\left(\sqrt{\pi}\right)^n}\left(\sqrt{\pi}\right)^n = 1,\end{aligned}$$

where we have used the Gauss's integral

$$\int_{-\infty}^{+\infty} e^{-s^2}\mathrm{d}s = \sqrt{\pi}.$$

Similarly, the other equality from the enunciation can be proved. ■

In the following, we prove a result which generalizes the results from the Theorem 8.2.4

Theorem 8.2.5 *Consider Ω a bounded domain. If we denote by I_Ω the integral*

$$I_\Omega(t-\tau,x) = \int_\Omega V(t,\tau,x,\xi)\mathrm{d}\xi,$$

then for $x \in \Omega$, we have

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau,x) = 1,$$

the limit is still valid uniformly with regard to x, on compact sets from Ω, and, for $x \in \mathbb{R}^n \setminus \overline{\Omega}$

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau,x) = 0,$$

the limit is still valid uniformly with regard to x, on compact sets from $\mathbb{R}^n \setminus \overline{\Omega}$.

Proof Firstly, consider the case $x \in \Omega$. We use the notations

$$d_0 = \mathrm{dist}(x, \Omega),\ d_1 = \mathrm{dist}(Q, \partial\Omega),$$

where Q is a compact, arbitrarily fixed in Ω, such that $x \in Q$.

We remember that, by definition, we have

$$d_0 = \mathrm{dist}(x, \partial\Omega) = \sup_{y\in\partial\Omega} |x - y|,$$
$$d_1 = \mathrm{dist}(Q, \partial\Omega) = \sup_{y\in\partial\Omega, x\in Q} |x - y|.$$

Consider the balls $B(x, d_0)$ and $B(x, d_1)$ and then

$$B(x, d_1) \subset B(x, d_0) \subset \Omega. \tag{8.2.18}$$

Using the monotony of the integral and taking into account the inclusion (8.2.8), we will deduce

$$I_\Omega(t-\tau, x) = \int_\Omega V(t, \tau, x, \xi)\mathrm{d}\xi \geq$$
$$\geq \int_{B(x,d_0)} V(t, \tau, x, \xi)\mathrm{d}\xi \geq \int_{B(x,d_1)} V(t, \tau, x, \xi)\mathrm{d}\xi = \tag{8.2.19}$$
$$= \frac{1}{\left(2\sqrt{\pi}\right)^n \left(\sqrt{t-\tau}\right)^n} \int_{B(x,d_1)} \exp\left(-\frac{\sum\limits_{i=1}^{n}(x_i - \xi_i)^2}{4(t-\tau)}\right)\mathrm{d}\xi.$$

We make the change of variable

$$\xi_i - x_i = 2\sqrt{t-\tau}\eta_i,\ i = 1, 2, ..., n.$$

As in the proof of the Theorem 8.2.4, the value of the Jacobian of this change is

$$2^n \left(\sqrt{t-\tau}\right)^n.$$

With this change of variable, the last integral from Eq. (8.2.19), becomes:

$$\frac{1}{\left(\sqrt{\pi}\right)^n} \int_{B\left(0, \frac{d}{2\sqrt{t-\tau}}\right)} e^{-\sum\limits_{i=1}^{n}\eta_i^2}\mathrm{d}\eta, \tag{8.2.20}$$

in which

$$\sqrt{\sum_{i=1}^{n}(\xi_i - x_i)^2} = 2\sqrt{t-\tau}\sqrt{\sum_{i=1}^{n}\eta_i^2}.$$

If we pass to the limit in Eq. (8.2.19) with $t-\tau \to 0^+$ and we take into account Eq. (8.2.20), we obtain

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \geq \lim_{t-\tau\to 0^+} \frac{1}{\left(\sqrt{\pi}\right)^n}\int_{B(0,\frac{d}{2\sqrt{t-\tau}})} e^{-\sum\limits_{i=1}^{n}\eta_i^2} d\eta =$$
$$= \frac{1}{\left(\sqrt{\pi}\right)^n}\int_{\mathbb{R}^n} e^{-\sum\limits_{i=1}^{n}\eta_i^2} d\eta = 1,$$

in which we have used the Gauss's integral. Also, we have used the fact that for $t-\tau \to 0^+$, we have

$$\frac{d}{2\sqrt{t-\tau}} \to \infty$$

and then the ball

$$B(0, \frac{d}{2\sqrt{t-\tau}})$$

becomes the whole space $\mathbb{R}^n$. Also, we already proved that

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \geq 1. \tag{8.2.21}$$

Since $\Omega \subset \mathbb{R}^n$, we have, obviously, that

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \leq \int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi = 1,$$

and then

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \leq 1. \tag{8.2.22}$$

From Eqs. (8.2.21) and (8.2.22), the first part of the proof is concluded. The limit takes place uniformly with regard to x, on compact sets from Ω that contains x, since d used in the above considerations depends only on the compact set that contains x, not on the choosing of x in the respective compact set.

Now, we approach the case when $x \in \mathbb{R}^n \setminus \overline{\Omega}$. Taking into account that Ω has been assumed to be domain (bounded), using the Jordan's theorem, we have that $\mathbb{R}^n \setminus \overline{\Omega}$

is a domain too. Let us consider a compact set $Q^* \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that $x \in Q^*$ and consider the distances $d_0^* = \mathrm{dist}(x, \partial\Omega)$, $d_1^* = \mathrm{dist}(Q^*, \partial\Omega)$ and the balls $B(x, d_0^*)$ and $B(x, d_0^*)$. Since $d_0^* > d_1^*$, we will deduce

$$B(x, d_1^*) \subset B(x, d_0^*) \Rightarrow$$
$$\Rightarrow \Omega \subset \mathbb{R}^n \setminus B(x, d_0^*) \subset \mathbb{R}^n \setminus B(x, d_1^*).$$

Corresponding to $I_\Omega(t-\tau, x)$ we have the evaluations

$$\begin{aligned} 0 \le I_\Omega(t-\tau, x) = \int_\Omega V(t, \tau, x, \xi)\mathrm{d}\xi \le \\ \le \int_{\mathbb{R}^n \setminus B(x, d_0^*)} V(t, \tau, x, \xi)\mathrm{d}\xi \le \int_{\mathbb{R}^n \setminus B(x, d_1^*)} V(t, \tau, x, \xi)\mathrm{d}\xi. \end{aligned} \tag{8.2.23}$$

We make the change of variable

$$\xi_i - x_i = 2\sqrt{t-\tau}\eta_i,\ i = 1, 2, ..., n.$$

Based on the considerations from the first part of the proof, the last integral from Eq. (8.2.23) becomes

$$\frac{1}{\left(\sqrt{\pi}\right)^n} \int_{\mathcal{D}} e^{-\sum\limits_{i=1}^{n} \eta_i^2} d\eta, \tag{8.2.24}$$

where the domain of integration $\mathcal{D}$ is

$$\mathcal{D} = R^n \setminus B(0, \frac{d^*}{2\sqrt{t-\tau}}).$$

On taking the limit for $t - \tau \to 0^+$, the radius

$$\frac{d^*}{2\sqrt{t-\tau}}$$

becomes infinite and then the ball

$$B(0, \frac{d^*}{2\sqrt{t-\tau}})$$

becomes the whole space $\mathbb{R}^n$. Therefore, the integral from Eq. (8.2.24) tends to zero and, comes back to Eq. (8.2.23), it results

$$0 \le \lim_{t-\tau \to 0^+} I_\Omega(t-\tau, x) \le \int_{\mathcal{D}} e^{-\sum\limits_{i=1}^{n} \eta_i^2} d\eta = 0,$$

in which the domain of integration $\mathcal{D}$ is defined as below. The limit takes place uniformly with respect to x, on compact sets from Ω that contains x, since d^* used in the above considerations depends only on the compact set that contains x, not on the choosing of x in the respective compact set. ■

The usefulness of the results demonstrated in the Theorems 8.2.4 and 8.2.5 follows from the following theorem.

Theorem 8.2.6 *Consider Ω a bounded domain from $\mathbb{R}^n$ and we suppose that the function f is continuous and bounded on Ω.*

Then:

- *(i). If $x \in \Omega$,*

$$\lim_{t-\tau \to 0^+} \int_\Omega V(t, \tau, x, \xi) f(\xi) \mathrm{d}\xi = f(x),$$

  the limit taking place uniformly with respect to x, on compact sets from Ω.
- *(ii). If $x \in \mathbb{R}^n \setminus \Omega$,*

$$\lim_{t-\tau \to 0^+} \int_\Omega V(t, \tau, x, \xi) f(\xi) \mathrm{d}\xi = 0,$$

  the limit taking place uniformly with regard to x, on compact sets from $\mathbb{R}^n \setminus \Omega$.

Proof (i). Consider Q a compact set arbitrarily fixed, $Q \subset \Omega$, such that $x \in Q$. We have the evaluations

$$\begin{aligned}
\left| \int_\Omega V(t, \tau, x, \xi) f(\xi) \mathrm{d}\xi - f(x) \right| \leq \left| \int_\Omega V(t, \tau, x, \xi) f(\xi) \mathrm{d}\xi - \right.\\
\left. -f(x) \int_\Omega V(t, \tau, x, \xi) f(\xi) \mathrm{d}\xi \right| + \left| f(x) \int_\Omega V(t, \tau, x, \xi) \mathrm{d}\xi - f(x) \right| \leq \\
\leq \int_\Omega V(t, \tau, x, \xi) |f(x) - f(\xi)| \mathrm{d}\xi + |f(x)| \left| \int_\Omega V(t, \tau, x, \xi) \mathrm{d}\xi - 1 \right| \leq \\
\leq \int_{B(x,\delta)} V(t, \tau, x, \xi) |f(x) - f(\xi)| \mathrm{d}\xi + \int_{\mathbb{R}^n \setminus B(x,\delta)} V(t, \tau, x, \xi) |f(x) - f(\xi)| \mathrm{d}\xi + \\
+ c_0 \left| \int_\Omega V(t, \tau, x, \xi) \mathrm{d}\xi - 1 \right|,
\end{aligned} \tag{8.2.25}$$

where, by c_0 we have noted $c_0 = \sup_{x \in \Omega} f(x)$.

In order to use the continuity of the function f, we take, sufficient small, ε and then exists $\eta(\varepsilon)$ such that if

$$|x - \xi| < \eta(\varepsilon) \Rightarrow |f(x) - f(\xi)| < \varepsilon.$$

If in the evaluations from Eq. (8.2.25) we take $\delta < \eta(\varepsilon)$, it results that

$$\int_{B(x,\delta)} V(t,\tau,x,\xi)|f(x)-f(\xi)|\mathrm{d}\xi < \varepsilon \int_{B(x,\delta)} V(t,\tau,x,\xi)\mathrm{d}\xi \leq$$
$$\leq \varepsilon \int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi = \varepsilon.$$

Then

$$\int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)|f(x)-f(\xi)|\mathrm{d}\xi < 2c_0 \int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)\mathrm{d}\xi,$$

and

$$\lim_{t-\tau\to 0^+} \int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi = 0,$$

because $x \notin \Omega \setminus B(x,\delta)$ and then we can use the second part of the Theorem 8.2.5.

Finally, for the last integral from Eq. (8.2.25), we have

$$\lim_{t-\tau\to 0^+} \left| \int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi - 1 \right| = 0,$$

because $x \in \Omega$ and then we can use the first part of the Theorem 8.2.5. If we take into account these evaluations in Eq. (8.2.25), the point (i) is proved. We outline that the limit from (i) takes place uniformly with respect to x because the last integrals from Eq. (8.2.25) tend to zero, uniformly on compact sets from Ω.

(ii). We take, arbitrarily, a compact set Q^* such that $x \in Q^*$ and $Q^* \subset \mathbb{R}^n \setminus \overline{\Omega}$. Since, by hypothesis, f is a bounded function, we have

$$\left| \int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi \right| \leq \int_{\Omega} |V(t,\tau,x,\xi)|\,|f(\xi)|\,\mathrm{d}\xi \leq$$
$$\leq c_0 \int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi,$$

and then

$$0 \leq \lim_{t-\tau\to 0^+} \left| \int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi \right| \leq c_0 \int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi.$$

Since $x \notin \Omega$, based on the second part of the Theorem 8.2.5, these inequalities lead to the conclusion that

$$\lim_{t-\tau\to 0^+} \int_{\Omega} V(t,\tau,x,\xi)\mathrm{d}\xi = 0 \tag{8.2.26}$$

and then

$$\lim_{t-\tau\to 0^+}\left|\int_\Omega V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi\right| = 0,$$

the limit taking place uniformly with respect to x, on compact sets from $\mathbb{R}^n \setminus \overline{\Omega}$, because so has been obtained the limit from Eq. (8.2.26). ∎

In the following theorem, we give a generalization of the results from the Theorem 8.2.6.

Theorem 8.2.7 *Consider the function $g(\tau,\xi)$ assumed to be continuous and bounded on $\mathcal{T}_T \times \Omega$. If, more,*

$$\lim_{\tau\to t^+} g(\tau,\xi) = g(t,\xi)$$

the limit taking place uniformly with respect to ξ, on compact sets from Ω, then

– *(i). If $x \in \Omega$,*

$$\lim_{\tau\to t^-}\int_\Omega V(t,\tau,x,\xi)g(\tau,\xi)\mathrm{d}\xi = g(t,x),$$

the limit taking place uniformly with respect to x, on compact sets from Ω.

– *(ii). If $x \in \mathbb{R}^n \setminus \Omega$,*

$$\lim_{\tau\to t^-}\int_\Omega V(t,\tau,x,\xi)g(\tau,\xi)\mathrm{d}\xi = 0,$$

the limit taking place uniformly with respect to x, on compact sets from $\mathbb{R}^n \setminus \Omega$.

Proof (i). Consider Q a compact set arbitrarily fixed, $Q \subset \Omega$, such that $x \in Q \subset \Omega$. Then

$$\left|\int_\Omega V(t,\tau,x,\xi)g(\tau,\xi)\mathrm{d}\xi - g(t,x)\right| \le$$

$$\le\left|\int_\Omega V(t,\tau,x,\xi)[g(\tau,\xi)-g(t,\xi)]\mathrm{d}\xi\right| + \left|\int_\Omega V(t,\tau,x,\xi)\mathrm{d}\xi - g(t,x)\right|. \quad (8.2.27)$$

If in Eq. (8.2.27) we pass to the limit with $\tau \to t^-$, the first integral from the right-hand side tends to zero, based on the hypotheses, and the last integral from Eq. (8.2.27) tends to zero based on the Theorem 8.2.6. Also, we will deduce that both limits take place uniformly with respect to x, on compact sets from Ω, based on the hypotheses, and on the fact that so has been obtained the result from the Theorem 8.2.6.

(ii). In a similarly way, we can prove this result. ∎

8.3 Method of the Green's Function

Firstly, we will obtain the Green's formula for the heat equation. With that end in view, we define the operators $\mathcal{L}_{(\tau,\xi)}$ and $\mathcal{M}_{(\tau,\xi)}$ by

$$\begin{aligned}\mathcal{L}_{(\tau,\xi)}u &= \Delta_\xi u - \frac{\partial u}{\partial \tau},\\ \mathcal{M}_{(\tau,\xi)}v &= \Delta_\xi v + \frac{\partial v}{\partial \tau}.\end{aligned} \tag{8.3.1}$$

Consider Ω a bounded domain whose boundary $\partial\Omega$ admits a tangent plane, piecewise continuously varying.

In all that follows we will use the function $u(t, x)$ which satisfies the following standard hypotheses:

- $u \in C(\overline{\mathcal{T}_T} \times \overline{\Omega})$;
- $u_{x_i x_i},\ u_t \in C(\mathcal{T}_T \times \Omega)$, for $0 \le \tau < t \le T$.

If we amplify $(8.3.1)_1$ by $v(\tau, \xi)$ and $(8.3.1)_2$ by $u(\tau, \xi)$, we obtain

$$v\mathcal{L}u - u\mathcal{M}v = v\Delta_\xi u - u\Delta_\xi v - v\frac{\partial u}{\partial \tau} - u\frac{\partial v}{\partial \tau}$$

that is

$$v\mathcal{L}u - u\mathcal{M}v = v\Delta_\xi u - u\Delta_\xi v - \frac{\partial}{\partial \tau}(uv). \tag{8.3.2}$$

Proposition 8.3.1 *We suppose satisfied the above hypotheses with regard to the domain Ω and of the function u. If the function v satisfies the hypotheses of the function u, then there holds the Green's formula*

$$\begin{aligned}\int_\Omega \int_0^t [v\mathcal{L}u - u\mathcal{M}v]\mathrm{d}\tau\mathrm{d}\xi &= \int_{\partial\Omega}\int_0^t \left[v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right]\tau\mathrm{d}\sigma_\xi - \\ &- \int_\Omega u(t,\xi)v(t,\xi)\mathrm{d}\xi + \int_\Omega u(0,\xi)v(0,\xi)\mathrm{d}\xi.\end{aligned} \tag{8.3.3}$$

Proof We integrate the equality (8.3.2) on the set $\Omega \times [0, t]$:

$$\begin{aligned}\int_\Omega \int_0^t [v\mathcal{L}u - u\mathcal{M}v]\mathrm{d}\tau\mathrm{d}\xi &= \int_\Omega \int_0^t [v\Delta_\xi u - u\Delta_\xi v]\mathrm{d}\tau\mathrm{d}\xi - \\ &- \int_\Omega \int_0^t \frac{\partial}{\partial \tau}(uv)\,\mathrm{d}\tau\mathrm{d}\xi.\end{aligned} \tag{8.3.4}$$

By using the well known Gauss–Ostrogradski's formula, it results

$$\int_{\Omega}\int_0^t v\Delta_\xi u \mathrm{d}\tau\mathrm{d}\xi = \int_{\Omega}\int_0^t v\sum_{i=1}^n \frac{\partial^2 u}{\partial \xi_i^2}\mathrm{d}\tau\mathrm{d}\xi =$$

$$= \int_{\Omega}\int_0^t \sum_{i=1}^n v\frac{\partial}{\partial \xi_i}\left(\frac{\partial u}{\partial \xi_i}\right)\mathrm{d}\tau\mathrm{d}\xi = \int_{\partial\Omega}\int_0^t v\sum_{i=1}^n v\frac{\partial u}{\partial \xi_i}\cos\alpha_i \mathrm{d}\tau\mathrm{d}\sigma_\xi =$$

$$= \int_{\partial\Omega}\int_0^t v\frac{\partial u}{\partial \nu_\xi}\mathrm{d}\tau\mathrm{d}\sigma_\xi,$$

where ν is the outside unit normal to the surface $\partial\Omega$.

Similarly, we can obtain the following equality

$$\int_{\Omega}\int_0^t u\Delta_\xi v \mathrm{d}\tau\mathrm{d}\xi = \int_{\partial\Omega}\int_0^t u\frac{\partial v}{\partial \nu_\xi}\mathrm{d}\tau\mathrm{d}\sigma_\xi.$$

Then

$$\int_{\Omega}\int_0^t \frac{\partial}{\partial \tau}(uv)\,\mathrm{d}\tau\mathrm{d}\xi = \int_{\partial\Omega} uv|_0^t\,\mathrm{d}\xi =$$

$$= \int_{\partial\Omega}[u(t,\xi)v(t,\xi) - u(0,\xi)v(0,\xi)]\mathrm{d}\xi.$$

By using these evaluations in Eq. (8.3.4), we obtain the Green's formula. ■

The Green's formula (8.3.3) can be generalized in the sense that in the form (8.3.1) of the operators $\mathcal{L}$ and $\mathcal{M}$ instead of the Laplacean Δ we can take an arbitrarily linear operator of the order two.

So, we define, the operator L and its adjunct M by

$$Lu = \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

$$Mv = \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2(a_{ij}(x)v)}{\partial x_i\partial x_j} - \sum_{i=1}^n \frac{\partial(b_i(x)v)}{\partial x_i} + c(x)u, \tag{8.3.5}$$

where $a_{ij} = a_{ji} \in C^2(\Omega),\ b_i \in C^1(\Omega)$ and $c \in C^0(\Omega)$.

By using a similar procedure as in Eq. (8.2.1), we construct the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A}u = Lu - \frac{\partial u}{\partial t},$$

$$\mathcal{B}v = Mv + \frac{\partial v}{\partial t}. \tag{8.3.6}$$

Proposition 8.3.2 *We suppose to be satisfied the hypotheses from the Proposition 8.3.1 on the domain Ω and of the functions u and v. Moreover, we suppose that the operator L is elliptical. Then takes place the Green's formula:*

$$\int_\Omega\int_0^t [v\mathcal{A}u-u\mathcal{B}v]\mathrm{d}\tau\mathrm{d}\xi=\int_{\partial\Omega}\int_0^t\left\{\gamma\left[v\frac{\partial u}{\partial\gamma}-u\frac{\partial v}{\partial\gamma}\right]+buv\right\}\mathrm{d}\tau\mathrm{d}\sigma_\xi-$$
$$-\int_\Omega u(t,\xi)v(t,\xi)\mathrm{d}\xi+\int_\Omega u(0,\xi)v(0,\xi)\mathrm{d}\xi. \tag{8.3.7}$$

Proof We multiply Eq. $(8.3.6)_1$ by v and Eq. $(8.3.6)_2$ by u and subtract the resulting relations, whence it follows the equality:

$$v\mathcal{A}u-u\mathcal{B}v=vLu-uMv-v\frac{\partial u}{\partial t}-u\frac{\partial v}{\partial t}=$$
$$=vLu-uMv-\frac{\partial}{\partial t}(uv)\,.$$

We integrate this equality on the set $\Omega\times[0,t]$ and, after we use the Gauss-Ostrogradski's formula, we are led to the Green's formula (8.3.7). ■

Consider, again, the operators $\mathcal{L}$ and $\mathcal{M}$ defined in Eq. (8.3.1). Corresponding, we will use the Green's formula in the form (8.3.3). Starting from this form of the Green's formula, we intend to find the form of the Riemann–Green's formula. In this hope, we use, again, the function $V(t,\tau,x,\xi)$ defined by

$$V(t,\tau,x,\xi)=\frac{1}{\left(2\sqrt{\pi}\right)^n\left(\sqrt{t-\tau}\right)^n}\exp\left(-\frac{\sum\limits_{i=1}^n(x_i-\xi_i)^2}{4(t-\tau)}\right). \tag{8.3.8}$$

The unique singular point of the function $V(t,\tau,x,\xi)$ is the point $(t,x)=(\tau,\xi)$. To avoid this point, we will consider the domain

$$\{\tau;\ 0\le\tau\le t-\delta,\ \delta>0\}\times\Omega.$$

On this domain we write the Green's formula (8.3.3) for the pair of functions (v,u), where $v=V(t,\tau,x,\xi)$ and $u=u(\tau,\xi)$:

$$\int_\Omega\int_0^{t-\delta}[V(t,\tau,x,\xi)\mathcal{L}u(\tau,\xi)-u(\tau,\xi)\mathcal{M}V(t,\tau,x,\xi)]\,\mathrm{d}\tau\mathrm{d}\xi=$$
$$=\int_{\partial\Omega}\int_0^{t-\delta}\left[V(t,\tau,x,\xi)\frac{\partial u}{\partial\nu}(\tau,\xi)-u(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi-$$
$$-\int_\Omega V(t,t-\tau,x,\xi)u(t-\tau,\xi)\mathrm{d}\xi+\int_\Omega V(t,0,x,\xi)u(0,\xi)\mathrm{d}\xi. \tag{8.3.9}$$

In this equality we pass to the limit with $\delta \to 0$ and we use the Theorem 8.2.7 from the Sect. 8.2. Thus, if $x \in \Omega$, it results

$$u(t,x) = -\int_\Omega \int_0^t V(t,\tau,x,\xi)\mathcal{L}u(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi +$$
$$+\int_{\partial\Omega}\int_0^t \left[V(t,\tau,x,\xi)\frac{\partial u}{\partial \nu}(\tau,\xi) - u(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial \nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi +$$
$$+\int_\Omega V(t,0,x,\xi)u(0,\xi)\mathrm{d}\xi. \tag{8.3.10}$$

The result proved here will be synthesized in the following theorem.

Theorem 8.3.1 *For the heat equation, the Riemann-Green's formula has the form of Eq. (8.3.10), where the operators $\mathcal{L}$ and $\mathcal{M}$ are defined in Eq. (8.3.1) and $V(t,\tau,x,\xi)$ is a function that has the form of Eq. (8.3.8).*

Remark. If $x \in \mathbb{R}^n \setminus \overline{\Omega}$ then by taking the limit in Eq. (8.3.9) with $\delta \to 0$ and by using the second part of the Theorem 8.2.7 (Sect. 8.2), it results:

$$0 = -\int_\Omega \int_0^t V(t,\tau,x,\xi)\mathcal{L}u(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi +$$
$$+\int_{\partial\Omega}\int_0^t \left[V(t,\tau,x,\xi)\frac{\partial u}{\partial \nu}(\tau,\xi) - u(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial \nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi +$$
$$+\int_\Omega V(t,0,x,\xi)u(0,\xi)\mathrm{d}\xi.$$

Now, we consider the initial boundary values problem

$$\begin{aligned}
&\mathcal{L}u(t,x) = f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \Omega,\\
&u(0,x) = \varphi(x),\ \forall x \in \overline{\Omega},\\
&u(t,y) = \alpha(t,y),\ \forall (t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega,\\
&\frac{\partial u}{\partial \nu}u(t,y) = \beta(t,y),\ \forall (t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega.
\end{aligned}$$

Then, the Riemann-Green's formula receives the form

$$u(t,x) = -\int_\Omega \int_0^t V(t,\tau,x,\xi) f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi +$$
$$+\int_{\partial\Omega}\int_0^t V(t,\tau,x,\xi)\beta(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi - \int_{\partial\Omega}\int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial \nu}\alpha(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi +$$
$$+\int_\Omega V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi. \tag{8.3.11}$$

The integrals from the right-hand side of the formula (8.3.11) are the associated potentials of the heat problem, namely,:

$$I_1 = -\int_{\Omega}\int_0^t V(t,\tau,x,\xi) f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi$$

is the heat potential of the volume;

$$I_2 = \int_{\partial\Omega}\int_0^t V(t,\tau,x,\xi)\beta(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi$$

is the surface heat potential of the simple layer;

$$I_3 = -\int_{\partial\Omega}\int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\alpha(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi$$

is the surface heat potential of the double layer;

$$I_4 = \int_{\Omega} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi$$

is the heat temporal potential.

Therefore, formula (8.3.11) is also called the formula of the heat potentials. As in the case of the elliptical equations, the heat potentials are used to solve the initial boundary value problems, in the context of the parabolical equations.

More exactly, the heat potentials permit the transformation of these problems in integral equations of the Fredholm type.

Let us consider the problem of Dirichlet type

$$\begin{aligned}
\Delta_\xi u(\tau,\xi) - \frac{\partial u}{\partial\tau}(\tau,\xi) &= f(\tau,\xi),\ \forall(\tau,\xi)\in\mathcal{T}_T\times\Omega,\\
u(0,\xi) &= \varphi(\xi),\ \forall\xi\in\overline{\Omega},\\
u(\tau,\eta) &= \alpha(\tau,\eta),\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega,
\end{aligned}\tag{8.3.12}$$

where Ω is a bounded domain with the boundary $\partial\Omega$ having the tangent plane piecewise continuously varying. We denote by $\mathcal{T}_T$ the interval $(0, T]$ and by $\overline{\mathcal{T}_T}$ the closed interval $[0, T]$. The functions f, φ and α are given and continuous on the indicated domains. The condition $(8.3.12)_3$ is called the Dirichlet condition. In a problem of Neumann type, the condition $(8.3.12)_3$ is replaced by the boundary condition of Neumann type

$$\frac{\partial u}{\partial\nu}(\tau,\eta) = \beta(\tau,\eta),\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega.$$

Definition 8.3.1 We call the Green's function attached to the domain Ω, to the operator $\mathcal{L}$ and to the Dirichlet condition (8.3.12)$_3$, the function $G(t,\tau,x,\xi)$ defined by

$$G(t,\tau,x,\xi) = V(t,\tau,x,\xi) + g(t,\tau,x,\xi), \tag{8.3.13}$$

where the function $V(t,\tau,x,\xi)$ is defined in Eq. (8.3.8) and $g(t,\tau,x,\xi)$ has the properties:

- $g(t,\tau,x,\xi)$ is a continuous function with respect to the variables t,τ,x and ξ on the set $\overline{\mathcal{T}_T} \times \overline{\mathcal{T}_T} \times \Omega \times \Omega$;
- the derivatives $g_{x_i x_i}$ and g_t are continuous functions on the set $\mathcal{T}_T \times \mathcal{T}_T \times \Omega \times \Omega$;
- $g(t,\tau,x,\xi)$ satisfies the adjunct homogeneous equation of the heat

$$\mathcal{M}g(t,\tau,x,\xi) = \Delta_\xi g(t,\tau,x,\xi) + \frac{\partial}{\partial\tau} g(t,\tau,x,\xi) = 0;$$

- $g(t,\tau,x,\xi)$ satisfies the condition $g(t,t,x,\xi) = 0$.

The Green's function $G(t,\tau,x,\xi)$ satisfies, by definition, the homogeneous Dirichlet condition

$$G(t,\tau,x,\eta) = 0, \ \forall(\tau,\eta) \in \overline{\mathcal{T}_T} \times \partial\Omega.$$

In the following theorem we prove that if the Dirichlet problem (8.3.12) admits a classical solution, then this solution can be represented with the aid of the Green's function.

Theorem 8.3.2 *If we suppose that the Dirichlet problem (8.3.12) admits a classical solution, then it has the form*

$$u(t,x) = -\int_\Omega \int_0^t G(t,\tau,x,\xi) f(\tau,\xi) \mathrm{d}\tau \mathrm{d}\xi -$$
$$-\int_{\partial\Omega} \int_0^t \frac{\partial G(t,\tau,x,\eta)}{\partial\nu} \alpha(\tau,\eta) \mathrm{d}\tau \mathrm{d}\sigma_\eta + \int_\Omega G(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi. \tag{8.3.14}$$

Proof We write the Green's formula (8.3.7) for the pair of functions $v = g(t,\tau,x,\xi)$ and $u = u(\tau,\xi)$, where $u(\tau,\xi)$ is the solution of the problem (8.3.12):

$$0 = -\int_\Omega \int_0^t g(t,\tau,x,\xi) f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi + \int_\Omega \int_0^t u(\tau,\xi)\mathcal{M}g(t,\tau,x,\xi)\mathrm{d}\tau\mathrm{d}\xi +$$
$$+ \int_{\partial\Omega} \int_0^t \left[g(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu} - u(\tau,\xi)\frac{\partial g(t,\tau,x,\xi)}{\partial\nu}\right] \mathrm{d}\tau\mathrm{d}\sigma_\xi -$$
$$- \int_\Omega g(t,\tau,x,\xi)u(\tau,\xi)\mathrm{d}\xi + \int_\Omega g(t,0,x,\xi)\varphi(\tau,\xi)\mathrm{d}\xi.$$

Based on the hypotheses imposed to the function g, this equality becomes

$$0=-\int_\Omega\int_0^t g(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi+\int_\Omega g(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi+$$
$$+\int_{\partial\Omega}\int_0^t\left[g(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu}-\alpha(\tau,\xi)\frac{\partial g(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi. \quad (8.3.15)$$

Now, we write the Riemann-Green's formula (8.3.10) for the pair of functions $v=V(t,\tau,x,\xi)$ and $u=u(\tau,\xi)$, where $u(\tau,\xi)$ is the solution of the problem (8.3.12):

$$u(t,x)=-\int_\Omega\int_0^t V(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi+\int_\Omega V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi+$$
$$+\int_{\partial\Omega}\int_0^t\left[V(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu}-\alpha(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi. \quad (8.3.16)$$

By adding, term by term the formulas (8.3.15) and (8.3.16), it results

$$u(t,x)=-\int_\Omega\int_0^t G(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi+$$
$$+\int_{\partial\Omega}\int_0^t G(t,\tau,x,\xi)\frac{u(\tau,\xi)}{\partial\nu}\mathrm{d}\tau\mathrm{d}\sigma_\xi-\int_{\partial\Omega}\int_0^t\frac{\partial G(t,\tau,x,\xi)}{\partial\nu}\alpha(\tau,\eta)\mathrm{d}\tau\mathrm{d}\sigma_\eta+$$
$$+\int_\Omega G(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi.$$

since the Green's function $G(t,\tau,x,\xi)$ becomes null on the boundary (because, by definition, $G(t,\tau,x,\xi)$ satisfies the homogeneous Dirichlet's condition), we will deduce that the second integral from the right-hand side of the above equality disappears, and the remained formula is even (8.3.14). ■

In the following, we make now analogous considerations for the Neumann's problem which can be deduced from the Dirichlet's problem (8.3.12) by substituting the condition $(8.3.12)_3$ with the condition

$$\frac{\partial u(\tau,\eta)}{\partial\nu}=\beta(\tau,\eta),\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega. \quad (8.3.17)$$

The Green's function for the domain Ω, the operator $\mathcal{L}$ and the Neumann's condition (8.3.17) is given in the formula (8.3.13) from the Definition 8.3.1, but the last condition from this definition is replaced by

$$\frac{\partial G(t,\tau,x,\eta)}{\partial\nu}=0,\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega, \quad (8.3.18)$$

that is, the function G satisfies the Neumann's condition in its homogeneous form.

Proposition 8.3.3 *We suppose that the Neumann's problem* $(8.3.12)_1$, $(8.3.12)_2$ *and* (8.3.17) *admits a classical solution. Then, it can be expressed with the aid of the Green's function in the form*

$$u(t,x) = -\int_{\Omega}\int_0^t G(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi +$$
$$+\int_{\partial\Omega}\int_0^t G(t,\tau,x,\eta)\beta(\tau,\eta)\mathrm{d}\tau\mathrm{d}\sigma_\eta + \int_{\Omega} G(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi. \quad (8.3.19)$$

Proof We will use the same reasoning as in the proof of the formula (8.3.14). Firstly, we write the Green's formula for the pair of functions $v = g(t,\tau,x,\xi)$ and $u = u(\tau,\xi)$, where $u(\tau,\xi)$ is the solution of our Neumann's problem. Then, we write the Riemann-Green's formula for the pair of functions $v = V(t,\tau,x,\xi)$ and $u = u(\tau,\xi)$, where $u(\tau,\xi)$ is the solution of the Neumann's problem. By adding, term by term the two resulting relations and take into account the conditions imposed to the functions $g(t,\tau,x,\xi)$, $G(t,\tau,x,\xi)$, we obtain the formula (8.3.19). ■

If we examine the formulas (8.3.14) and (8.3.19), we are led to the conclusion that the solutions of the Dirichlet problem and Neumann's problem, if exists, can be unique represented with the aid of the Green's function. Since the function $V(t,\tau,x,\xi)$, from the definition of the Green's function, is defined in Eq. (8.3.8), it results that in view of determination of the Green's function we must determine the function $g(t,\tau,x,\xi)$. Apparently, the problem of determining the function $g(t,\tau,x,\xi)$ has the same difficulty like the proper problem of determining the solution of the Dirichlet's problem or Neumann's problem, especially, on account to the conditions of regularity imposed to the function $g(t,\tau,x,\xi)$.

But unlike to the classical solution u, the function $g(t,\tau,x,\xi)$ satisfies in the case of the Dirichlet's problem and in the case of the Neumann's problem, an homogeneous equation of heat. Also, in the problem of Dirichlet as well as the problem of Neumann, the solution u satisfies a boundary condition with α respectively, β arbitrarily. In the present case, the function $g(t,\tau,x,\xi)$ satisfies a boundary condition in which the right-hand side is perfectly determined, because

$$g(t,\tau,x,\eta) = -V(t,\tau,x,\eta),\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega,$$

and, respectively

$$\frac{\partial g(t,\tau,x,\eta)}{\partial\nu} = -\frac{\partial V(t,\tau,x,\eta)}{\partial\nu},\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega,$$

where $V(t,\tau,x,\eta)$ is given in Eq. (8.3.8).

These comments prove that the method of the Green's function can be successfully used to find the solution of the initial-boundary values problems from the theory of the parabolical equations.

In the considerations from this paragraph, the method of the Green's function has been used to find the solution of the linear problems. But this method can be used to find the solution of the nonlinear problems. We want outline that for the determination of the Green's function one can use the Laplace's transform. Applying the Laplace's transform on the parabolical equations and on the initial and boundary conditions we obtain an elliptical boundary value problem, because the Laplace's transform proceeds on the temporal variable. Also, an initial-boundary value problem for parabolical equations, receives some simplifications if we apply the Fourier's transform on the spatial variables.

Let us consider the nonlinear problem

$$\begin{gathered}\Delta u-\frac{\partial u}{\partial t}=F(t,x,u,u_{x_1},u_{x_2},...,u_{x_n}),\ \forall(t,x)\in\mathcal{T}_T\times\Omega,\\ u(0,x)=\varphi(x),\ \forall x\in\overline{\Omega},\\ u(t,y)=\alpha(t,y),\ \forall(t,y)\in\overline{\mathcal{T}_T}\times\partial\Omega.\end{gathered}\tag{8.3.20}$$

In tackling of the problem (8.3.20) we can use the same procedure as in the case of the linear problems. Firstly, we determine the Green's function attached to the domain Ω, to the linear operator $\Delta u-u_t$ and to the boundary conditions $(8.3.20)_3$. Assuming that the problem (8.3.20) admits a classical solution, then this solution can be represented with the aid of the Green's function in the form

$$\begin{aligned}u(t,x)=&-\int_{\Omega}\int_0^t G(t,\tau,x,\xi)F(\tau,\xi,u,u_{\xi_1},u_{\xi_2},...,u_{\xi_n})\mathrm{d}\tau\mathrm{d}\xi-\\ &-\int_{\partial\Omega}\int_0^t\frac{\partial G(t,\tau,x,\eta)}{\partial\nu}\alpha(\tau,\eta)\mathrm{d}\tau\mathrm{d}\sigma_\eta+\int_{\Omega}G(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi.\end{aligned}\tag{8.3.21}$$

We must now determine the conditions that will be imposed to the functions F, α and φ such that the function u from Eq. (8.3.21) is an effective solution for the problem (8.3.20). One can prove a result of the form: If the function F is continuous in all its variables and satisfies a Lipschitz condition in the variables $u, u_{x_1}, u_{x_2}, ..., u_{x_n}$, then u from Eq. (8.3.21) is an effective solution of the problem (8.3.20).

8.4 Cauchy's Problem

In the considered initial-boundary values problems for the heat equation, in the previous paragraphs, it is essential to know the temperature on the surface of the body where the problem has been stated.

In this paragraph we consider that the surface is to a great distance, such that instead of a bounded domain we will consider the whole space $\mathbb{R}^n$. Therefore, the boundary conditions disappear and then we have the following Cauchy's problem

$$\Delta u(t,x) - u_t(t,x) = f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n,$$
$$u(0,x) = \varphi(x),\ \forall x \in \mathbb{R}^n, \tag{8.4.1}$$

where $\mathcal{T}_T$ is a temporal interval $(0,T]$ and the functions f and φ are given and continuous on $\mathcal{T}_T \times \mathbb{R}^n$, respectively, on $\mathbb{R}^n$.
The problem (8.4.1) will be complete if it is known the behavior of the function u to the infinity. It is well known two kind of behavior to the infinity:

- the function u is bounded;
- u asymptotically tends to zero.

In all that follows we will suppose that u is bounded to the infinity.

We call *the classical solution* for the Cauchy's problem, a function u which satisfies the conditions:

- $u \in C(\overline{\mathcal{T}_T} \times \mathbb{R}^n)$;
- u and u_{x_i} are bounded functions on $\mathcal{T}_T \times \mathbb{R}^n$;
- $u_{x_i x_i},\ u_t \in C(\mathcal{T}_T \times \mathbb{R}^n)$;
- u satisfies the Eq. $(8.4.1)_1$ and the initial condition $(8.4.1)_2$.

In tackling of the Cauchy's problem (8.4.1), we will make two steps. First, assuming that the problem admits a classical solution, we will find its form with the aid of the Riemann-Green's formula.

In the second step, we will show that in certain conditions of regularity imposed to the functions f and φ the founded formula for u is an effective solution for the problem (8.4.1).

We remember that the fundamental solution $V(t,\tau,x,\xi)$ is given by

$$V(t,\tau,x,\xi) = \frac{1}{\left(2\sqrt{\pi}\right)^n \left(\sqrt{t-\tau}\right)^n} \exp\left(-\frac{\sum\limits_{i=1}^{n}(x_i-\xi_i)^2}{4(t-\tau)}\right). \tag{8.4.2}$$

Theorem 8.4.1 *We suppose that the Cauchy's problem (8.4.1) admits a classical solution. Then, this solution admits the representation:*

$$u(t,x) = -\int_{\mathbb{R}^n}\int_0^t V(t,\tau,x,\xi) f(\tau,\xi)\,\mathrm{d}\tau\mathrm{d}\xi + \int_{\mathbb{R}^n} V(t,0,x,\xi)\varphi(\xi)\,\mathrm{d}\xi. \tag{8.4.3}$$

Proof We arbitrarily fix $x \in \mathbb{R}^n$ and we take the ball $B(0,R)$ with the center in the origin and the radius R sufficient big such that the ball contains inside the point x. We write then the Riemann-Green's formula on this ball, for the pairs of functions $v = V(t,\tau,x,\xi)$ and $u = u(t,x)$, where $u(t,x)$ is the solution of the problem (8.4.1)

$$u(t,x) = -\int_{B(0,R)}\int_0^t V(t,\tau,x,\xi) f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi + \int_{B(0,R)} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi +$$
$$+\int_{\partial B(0,R)}\int_0^t \left[V(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu} - u(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi. \tag{8.4.4}$$

We have assumed that u and u_{x_i} are bounded functions (because u is a classical solution for the problem (8.4.1)). Then, taking into account the properties of the function $V(t,\tau,x,\xi)$, we can show that if $R \to \infty$, the last integral from Eq. (8.4.4) tends to zero. With that end in view we write the last integral from Eq. (8.4.4) in the form

$$\int_{\partial B(0,R)}\int_0^t V(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu}\mathrm{d}\tau\mathrm{d}\sigma_\xi -$$
$$-\int_{\partial B(0,R)}\int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial\nu}u(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi = I_1 + I_2. \tag{8.4.5}$$

Then

$$|I_1| \le c_0 \int_{\partial B(0,R)}\int_0^t \frac{1}{\left(\sqrt{t-\tau}\right)^n}\exp\left(-\frac{\sum\limits_{i=1}^n (x_i-\xi_i)^2}{4(t-\tau)}\right)\mathrm{d}\tau\mathrm{d}\sigma_\xi,$$

where

$$c_0 = \frac{1}{\left(2\sqrt{\pi}\right)^n}\sup\frac{\partial u}{\partial\nu}$$

and this supremum exists since u is a bounded function.

It is clear that

$$|x_k - \xi_k| \le r = |\xi x| = \sqrt{\sum_{i=1}^n (x_i-\xi_i)^2}.$$

We can choose the radius of the ball R such that for x arbitrarily fixed, $x \in Int B(0,R)$ and $\xi \in \partial B(0,R)$, we have $|\xi x| > R/2$.

By using these evaluations, for I_1 we obtain

$$|I_1| \le c_0 \int_{\partial B(0,R)}\int_0^t \frac{1}{\left(\sqrt{t-\tau}\right)^n} e^{-\frac{R^2}{16(t-\tau)}}\mathrm{d}\tau\mathrm{d}\sigma_\xi.$$

With regard to the derivative of the function $V(t, \tau, x, \xi)$ in the direction of the normal, we have the estimation

$$\left|\frac{\partial V}{\partial \nu}\right| = \left|\sum_{k=1}^{n} \frac{\partial V}{\partial x_k} \cos \alpha_k\right| \leq \left|\sum_{k=1}^{n} \frac{\partial V}{\partial x_k}\right|,$$

Therefore, for I_2 we obtain

$$|I_2| \leq c_1 \int_{\partial B(0,R)} \int_0^t \sum_{i=1}^{n} |x_i - \xi_i| \, (t-\tau)^{-\frac{n+2}{2}} \exp\left(-\frac{\sum\limits_{i=1}^{n} (x_i - \xi_i)^2}{4(t-\tau)}\right) \mathrm{d}\tau \mathrm{d}\sigma_\xi,$$

and with the above estimations, it results

$$|I_2| \leq c_2 \int_{\partial B(0,R)} \int_0^t \frac{r}{(t-\tau)^{(n+2)/2}} e^{-\frac{r^2}{4(t-\tau)}} \mathrm{d}\tau \mathrm{d}\sigma_\xi \leq$$
$$\leq c_2 R \int_{\partial B(0,R)} \int_0^t \frac{1}{(t-\tau)^{(n+2)/2}} e^{-\frac{R^2}{16(t-\tau)}} \mathrm{d}\tau \mathrm{d}\sigma_\xi,$$

where c_1 proceeds from the supremum of the function u and $c_2 = nc_1$.

If we make the change of variable

$$t - \tau = \frac{R^2}{16\sigma^2} \Rightarrow \mathrm{d}\tau = \frac{R^2}{8\sigma^3} \mathrm{d}\sigma,$$

then for the increase of I_2 we have

$$|I_2| \leq c_3 \frac{1}{R^{n-1}} \int_{\partial B(0,R)} \int_{\frac{R}{4\sqrt{t}}}^{\infty} \sigma^{n-1} e^{-\sigma^2} \mathrm{d}\sigma \mathrm{d}\sigma_\xi =$$
$$= c_3 \omega_n \int_{\frac{R}{4\sqrt{t}}}^{\infty} \sigma^{n-1} e^{-\sigma^2} \mathrm{d}\sigma \mathrm{d}\sigma_\xi.$$

An analogous increase follows for I_1 too, by using the same change of variable. By integrating $n-1$-times by parts, it will be shown that

$$\lim_{R \to \infty} \int_{\frac{R}{4\sqrt{t}}}^{\infty} \sigma^{n-1} e^{-\sigma^2} \mathrm{d}\sigma \mathrm{d}\sigma_\xi = 0.$$

Therefore, I_1 and I_2 tend to zero, for $R \to \infty$. If we pass to the limit with $R \to \infty$ in Eq. (8.4.5), we obtain that the integral from left-hand side tends to zero. So, if we pass to the limit with $R \to \infty$ in Eq. (8.4.4), we obtain formula (8.4.3). ∎

The formula (8.4.3) is called *the Poisson's formula* for the representation of the solution of the Cauchy's problem (8.4.1). With the aid of the Poisson's formula we can prove the uniqueness of the classical solution for the problem (8.4.1).

Theorem 8.4.2 *The Cauchy's problem (8.4.1) admits at the most one classical solution.*

Proof We suppose, through absurd, that the problem (8.4.1) admits two classical bounded solutions, $u_1(t,x)$ and $u_2(t,x)$, that is,

$$\begin{aligned}\Delta u_i(t,x) - \frac{\partial u_i}{\partial t}(t,x) &= f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n,\\ u_i(0,x) &= \varphi(x),\ \forall x \in \mathbb{R}^n,\end{aligned}$$

where $i = 1, 2$. We define the function $v(t,x)$ by $v(t,x) = u_1(t,x) - u_2(t,x)$. Then

$$\begin{aligned}\Delta v(t,x) - \frac{\partial v}{\partial t}(t,x) &= f(t,x) - f(t,x) = 0,\\ v(0,x) = u_1(0,x) - u_2(0,x) &= \varphi(x) - \varphi(x) = 0.\end{aligned} \tag{8.4.6}$$

So, we have obtained a new Cauchy's problem with $f \equiv 0$ and $\varphi \equiv 0$. According to the Theorem 8.4.1, if a Cauchy's problem admits a solution, then the solution has indispensable the form (8.4.3). If we write formula (8.4.3) and take into account that $f \equiv 0$ and $\varphi \equiv 0$, then we obtain $v(t,x) = 0, ; \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n$ such that $u_1(t,x) = u_2(t,x)$. ■

It remains to prove that the function u from Eq. (8.4.3) is an effective solution of the Cauchy's problem (8.4.1). This is the objective of the following theorem of existence.

Theorem 8.4.3 *Assume the following conditions hold*

(i) the functions $f(t,x)$, $\frac{\partial f}{\partial x_i}(t,x)$, $\frac{\partial^2 f}{\partial x_i^2}(t,x)$ are continuous and bounded on $\mathcal{T}_T \times \mathbb{R}^n$, that is,

$$f(t,x),\ \frac{\partial f}{\partial x_i}(t,x),\ \frac{\partial^2 f}{\partial x_i^2}(t,x) \in C(\mathcal{T}_T \times \mathbb{R}^n) \cap B(\mathcal{T}_T \times \mathbb{R}^n);$$

(ii) the functions $\varphi(t,x)$, $\frac{\partial \varphi}{\partial x_i}(t,x)$, $\frac{\partial^2 \varphi}{\partial x_i^2}(t,x)$ are continuous and bounded on $\mathcal{T}_T \times \mathbb{R}^n$, that is,

$$\varphi(t,x),\ \frac{\partial \varphi}{\partial x_i}(t,x),\ \frac{\partial^2 \varphi}{\partial x_i^2}(t,x) \in C(\mathcal{T}_T \times \mathbb{R}^n) \cap B(\mathcal{T}_T \times \mathbb{R}^n).$$

Then, the function u from Eq. (8.4.3) is an effective solution of the Cauchy's problem (8.4.1), namely, a bounded solution on $\mathcal{T}_T \times \mathbb{R}^n$.

Proof We define the integral I_1 by

$$I_1 = \int_{\mathbb{R}^n} V(t, 0, x, \xi)\varphi(\xi)\mathrm{d}\xi,$$

and show that I_1 verifies the problem

$$\begin{aligned} \Delta u(t, x) - \frac{\partial u}{\partial t}(t, x) = 0,\ \forall (t, x) \in \mathcal{T}_T \times \mathbb{R}^n, \\ u(0, x) = \varphi(x),\ \forall x \in \mathbb{R}^n, \end{aligned} \tag{8.4.7}$$

and, also, we show that the integral I_2

$$I_2 = \int_{\mathbb{R}^n} \int_0^t V(t, \tau, x, \xi) f(\tau, \xi)\mathrm{d}\tau\mathrm{d}\xi,$$

verifies the problem

$$\begin{aligned} \Delta u(t, x) - \frac{\partial u}{\partial t}(t, x) = f(t, x),\ \forall (t, x) \in \mathcal{T}_T \times \mathbb{R}^n, \\ u(0, x) = 0,\ \forall x \in \mathbb{R}^n. \end{aligned}$$

Thus, it will be obvious that $I_1 + I_2$, that is, u from Eq. (8.4.3), verifies the Cauchy's problem (8.4.1).

Since φ is bounded and continuous, we have

$$|I_1| \leq \|\varphi\| \int_{\mathbb{R}^n} V(t, 0, x, \xi)\mathrm{d}\xi = \|\varphi\|,$$

that proves that the integral I_1 is convergent and, therefore, we can derive under the integral. Then

$$\Delta I_1 - \frac{\partial I_1}{\partial t} = \int_{\mathbb{R}^n} \left(\Delta V - \frac{\partial V}{\partial t}\right) \varphi(\xi)\mathrm{d}\xi = 0,$$

taking into account the properties of the function $V(t, \tau, x, \xi)$.

On the other hand, from the properties of the function $V(t, \tau, x, \xi)$, we have

$$\lim_{t\to 0} I_1 = \lim_{t\to 0} \lim_{R\to\infty} \int_{\mathbb{R}^n} V(t, 0, x, \xi)\varphi(\xi)\mathrm{d}\xi = \varphi(x).$$

Since the ball $B(0, R)$ has the radius R sufficient big, such that the point x is contained inside of the ball.

As in the case of I_1, it will be shown that the integral from I_2 is convergent and then we can derive under the integral such that

$$\Delta_x I_2 = -\int_0^t \int_{\mathbb{R}^n} \Delta_x V(t, \tau, x, \xi) f(\tau, \xi) \mathrm{d}\xi \mathrm{d}\tau. \tag{8.4.8}$$

In the case of the derivative with respect to t, we have an integral with parameter:

$$\frac{\partial I_2}{\partial t} = -\int_{\mathbb{R}^n} V(t, t, x, \xi) f(t, \xi) \mathrm{d}\xi - \int_{\mathbb{R}^n} \int_0^t \frac{\partial V(t, \tau, x, \xi)}{\partial \tau} f(\tau, \xi) \mathrm{d}\xi \mathrm{d}\tau. \tag{8.4.9}$$

For the first integral from the right-hand side of the relation (8.4.9) we have, in fact,

$$\lim_{\tau \to t^-} \lim_{R \to \infty} \int_{B(0,R)} V(t, \tau, x, \xi) f(\tau, \xi) \mathrm{d}\xi = f(t, x),$$

according to first part of the Theorem 8.2.7 (Sect. 8.2). Thus, Eq. (8.4.9) becomes

$$\frac{\partial I_2}{\partial t} = -f(t, x) - \int_{\mathbb{R}^n} \int_0^t \frac{\partial V(t, \tau, x, \xi)}{\partial \tau} f(\tau, \xi) \mathrm{d}\xi \mathrm{d}\tau,$$

relation which, together with Eq. (8.4.8), leads to

$$\Delta_x I_2 - \frac{\partial I_2}{\partial t} = f(t, x) -$$
$$- \int_{\mathbb{R}^n} \int_0^t \left[\Delta_x V(t, \tau, x, \xi) - \frac{\partial V(t, \tau, x, \xi)}{\partial \tau} \right] f(\tau, \xi) \mathrm{d}\tau \mathrm{d}\xi.$$

But

$$\Delta_x V(t, \tau, x, \xi) - \frac{\partial V(t, \tau, x, \xi)}{\partial \tau} = 0,$$

and then the previous relation becomes

$$\Delta_x I_2 - \frac{\partial I_2}{\partial t} = f(t, x).$$

Therefore, it is clear that

$$\lim_{t \to 0} I_2 = \int_0^0 \int_{\mathbb{R}^n} V(t, \tau, x, \xi) f(\tau, \xi) \mathrm{d}\xi \mathrm{d}\tau = 0,$$

that concludes the proof of the theorem. ■

At the end of this paragraph, we will solve a Cauchy's problem, attached to the equation of the heat, in a particular case.

Let $B = [0, T] \times (-\infty, \infty)$ be a band, where T is a positive fixed number, which can be ∞ too. Consider the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \ \forall (t, x) \in B. \tag{8.4.10}$$

If the function $u(t, x)$, defined on the band B, has the continuous derivatives $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ inside of the band and $u(t, x)$ satisfies the Eq. (8.4.10), we call that the function $u(t, x)$ is *a regular solution* of the Eq. (8.4.10).

The Cauchy's problem consists of the determination of a regular solution of the Eq. (8.4.10) which satisfies the initial condition:

$$u(0, x) = \varphi(x), \ \forall x \in (-\infty, -\infty), \tag{8.4.11}$$

where the function $\varphi(x)$ is a given real function, which is continuous and bounded.

We will prove that the function $u(t, x)$, defined by

$$u(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(\xi - x)^2}{4t}} \mathrm{d}\xi, \tag{8.4.12}$$

is the solution of the Cauchy's problem (8.4.10) and (8.4.11).

It is well known that the integral from Eq. (8.4.12) is uniformly convergent in a vicinity of an arbitrarily point (t, x) from the inside of the band B. If we make the change of variable $\xi = x + 2\eta\sqrt{t}$, formula (8.3.12) becomes:

$$u(t, x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x + 2\eta\sqrt{t}) e^{-\eta^2} d\eta. \tag{8.4.13}$$

Since φ is continuous and bounded, we have

$$\sup_{-\infty < x < \infty} |\varphi(x)| < M, \ M > 0.$$

The integral from Eq. (8.4.13) is absolute convergent and then

$$|u(t, x)| < \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = \frac{M}{\sqrt{\pi}} \sqrt{\pi} = M.$$

The integrals obtained by derivation under the integral in Eq. (8.4.12), with respect to x and with t, are uniformly convergent. On the other hand, the function

$$\frac{1}{\sqrt{t}} e^{-\frac{(\xi - x)^2}{4t}}, \ t > 0,$$

satisfies, obvious, the Eq. (8.4.10). These estimations assure that the function u defined in Eq. (8.4.12) satisfies the Eq. (8.4.10).

Using again the uniformly convergence of the integral in a vicinity of any point (t, x), with $t > 0$, from the inside of the band B, we can pass to the limit, as $t \to 0$, in Eq. (8.4.13), whence it follows

$$\lim_{t \to 0} u(t, x) = \varphi(x).$$

So, we immediately obtain the uniqueness and the stability of the regular solution for our Cauchy's problem. One can show that the regular solution of the Eq. (8.4.10), satisfies the inequality $m \le u(t, x) \le M$, where $m = \inf u(0, x)$ and $M = \sup u(0, x)$, $x \in (-\infty, \infty)$. One can then use the function $v(t, x) = 2t + x^2$, which obvious is a particular solution of the Eq. (8.4.10).

Chapter 9
Elliptic Partial Differential Equations

9.1 Introductory Formulas

Let us consider the three-dimensinal regular domain $D \subset R^3$ bounded by the Liapunov surface $S = \partial D$.

In the classical mathematical analysis the following formula is proved

$$\oint_S \left[P(x, y, z)\mathrm{d}y\mathrm{d}z + Q(x, y, z)\mathrm{d}z\mathrm{d}x + R(x, y, z)\right]\mathrm{d}x\mathrm{d}y = \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

which is called the *Gauss-Ostrogradski-Green's formula*.

In the following we deduce some particular forms of this formula, useful in the theory of elliptical equations.

Let φ be a scalar function, $\varphi = \varphi(x, y, z)$, $\varphi \in C^1(D)$, $D \subset R^3$. To this function we attach the differential operator, denoted by "grad"

$$\text{grad } \varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k}$$

called *the gradient of* φ. Now, consider a vectorial function $\vec{V} = \vec{V}(x, y, z)$, $\vec{V}\ (V_1, V_2, V_3)$ such that $V_i = V_i(x, y, z) \in C^1(D)$, $D \subset R^3$. To this vectorial function we can attach two differential operators namely, "*the divergence operator*", denoted "div" and "*the rotor operator*", denoted "curl", as follows

$$\text{div } \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z},$$

$$\text{curl } \vec{V} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\vec{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\vec{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\vec{k}.$$

M. Marin and A. Öchsner, *Complements of Higher Mathematics*,
https://doi.org/10.1007/978-3-319-74684-5_9

Therefore, if we consider the functions $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ as the components of a vectorial function

$$\vec{V}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

then the above Gauss-Ostrogradski-Green's formula can be restated in the form

$$\oint_S \vec{V}\vec{n}\mathrm{d}\sigma = \int_D \mathrm{div}\ \vec{V}\mathrm{d}v,\ \ \mathrm{d}v = \mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{9.1.1}$$

where $\vec{n}$ is the outward normal of the surface S that bounded the domain D, defined by

$$\vec{n} = (n_1, n_2, n_3) = (\cos\alpha, \cos\beta, \cos\gamma) = (\cos(x, \vec{n}), \cos(y, \vec{n}), \cos(z, \vec{n})).$$

For two scalar functions $\varphi(x, y, z)$ and $\psi(x, y, z)$ such that $\varphi, \psi \in C^2(D), D \subset R^3$, we can consider the vectorial function $\vec{V}$ defined by

$$\vec{V} = \varphi.\mathrm{grad}\ \psi.$$

Then, the scalar product $\vec{V}.\vec{n}$ becomes

$$\vec{V}.\vec{n} = \varphi.\mathrm{grad}\ \psi\ \vec{n} = \varphi.\frac{d\psi}{dn},$$

since

$$\frac{d\psi}{dn} = \frac{\partial\psi}{\partial x}n_1 + \frac{\partial\psi}{\partial y}n_2 + \frac{\partial\psi}{\partial z}n_3.$$

With these calculations, Eq. (9.1.1) becomes

$$\oint_S \varphi\frac{d\psi}{dn}\mathrm{d}\sigma = \int_D \mathrm{div}\ \ (\varphi\mathrm{grad}\ \psi)\,\mathrm{d}v. \tag{9.1.2}$$

On the other hand, if we use the notation

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k},$$

then the above defined operators "grad", "div" and "curl" can be written in the form

$$\mathrm{grad}\ \varphi = \nabla\varphi,\ \ \mathrm{div}\ \vec{V} = \nabla.\vec{V},\ \ \mathrm{curl}\ \vec{V} = \nabla \times \vec{V}.$$

Also, we have the following results

$$\nabla.(\varphi.\nabla\psi) = \nabla\varphi.\nabla\psi + \varphi\nabla(\nabla\psi) = \mathrm{grad}\ \varphi.\mathrm{grad}\ \psi + \varphi\Delta\psi,$$

because

$$\nabla(\nabla\psi) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \Delta\psi$$

Now, relation (9.1.2) can be restated as follows

$$\oint_S \varphi\frac{d\psi}{dn}\mathrm{d}\sigma = \int_D (\mathrm{grad}\ \varphi\mathrm{grad}\ \psi + \varphi\Delta\psi)\,\mathrm{d}v, \tag{9.1.3}$$

known as *the first Green's formula*.

For the vectorial function $\vec{V} = \psi\mathrm{grad}\ \varphi$ the first Green's formula becomes

$$\oint_S \psi\frac{d\varphi}{dn}\mathrm{d}\sigma = \int_D (\mathrm{grad}\ \psi\mathrm{grad}\ \varphi + \psi\Delta\varphi)\,\mathrm{d}v. \tag{9.1.4}$$

Subtracting, term by term, Eq. (9.1.4) from (9.1.3), we obtain

$$\oint_S \left(\varphi\frac{d\psi}{dn} - \psi\frac{d\varphi}{dn}\right)\mathrm{d}\sigma = \int_D (\varphi\Delta\psi - \psi\Delta\varphi)\,\mathrm{d}v, \tag{9.1.5}$$

known as *the second Green's formula*.

9.2 Potentials

Let us consider the scalar function $\varphi = \varphi(x, y, z)$ such that $\varphi \in C^2(D)$, $D \subset R^3$ and the domain Ω with the following properties:

(i) Ω is bounded by the regular surface $\Sigma = \partial\Omega$;
(ii) Σ has the continuous normal outward to the outside of Ω;
(iii) $\overline{\Omega} = \Omega \cup \Sigma$, $\overline{\Omega} \subset D$.
Since $\varphi \in C^2(D)$ and $\overline{\Omega} \subset D$ we deduce $\varphi \in C^2(\Omega)$.

Definition 9.2.1 By definition, the following integrals

$$\frac{1}{4\pi}\oint_\Sigma \frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma, \quad -\frac{1}{4\pi}\oint_\Sigma \varphi\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma, \quad -\frac{1}{4\pi}\int_\Omega \frac{1}{r}\Delta\varphi\mathrm{d}\sigma, \tag{9.2.1}$$

are called Single-Layer Potential of surface, Double-Layer Potential of surface and Volume Potential, respectively.

Here r is the Euclidean distance $r = \sqrt{x^2 + y^2 + z^2}$.

Theorem 9.2.1 *Let φ be a scalar function, $\varphi \in C^2(D)$, the domain Ω having the above properties and M_0 an arbitrary fixed point in Ω. Then, we have*

$$\varphi(M)=\varphi(x,y,z)=\frac{1}{4\pi}\oint_{\Sigma}\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma-\frac{1}{4\pi}\oint_{\Sigma}\varphi\frac{d}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma-\frac{1}{4\pi}\int_{\Omega}\frac{1}{r}\Delta\varphi\mathrm{d}v, \quad (9.2.2)$$
$$\forall M\in\Omega,\ where\ r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2},\ M_0=M_0\,(x_0,y_0,z_0)\,.$$

Proof Since $M\in\Omega$ and Ω is a domain we deduce that there exists a ball $B(M,\varrho)$ such that $\overline{B}(M,\varrho)\subset\Omega$, where $\overline{B}(M,\varrho)=B(M,\varrho)\cup S(M,\varrho)$. Here we have noted by $S(M,\varrho)$ the sphere with the center M and radius ϱ.

On the domain $\Omega\setminus\overline{B}(M,\varrho)$ we apply the second Green's formula (9.1.5) for the pair of functions φ and $\psi=1/r$:

$$\oint_{\partial(\Omega\setminus\overline{B})}\left(\varphi\frac{\mathrm{d}\psi}{\mathrm{d}n}-\psi\frac{\mathrm{d}\varphi}{\mathrm{d}n}\right)\mathrm{d}\sigma=\int_{\Omega\setminus\overline{B}}(\varphi\Delta\psi-\psi\Delta\varphi)\,\mathrm{d}v.$$

But $\partial\left(\Omega\setminus\overline{B}\right)=\Sigma\cup S$. Also, it is well known fact that

$$\Delta\psi=\Delta\frac{1}{r}=0,$$

such that the previous equality becomes

$$\oint_{\Sigma\cup S}\left(\varphi\frac{\mathrm{d}}{\mathrm{d}n}\frac{1}{r}-\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\right)\mathrm{d}\sigma=-\int_{\Omega\setminus\overline{B}}\frac{1}{r}\Delta\varphi\mathrm{d}v. \quad (9.2.3)$$

Let us make some evaluations on the left-hand side of Eq. (9.2.3), denoted by I_L. Firstly, we observe that

$$I_L=\oint_{\Sigma\cup S}\left(\varphi\frac{d}{dn}\frac{1}{r}-\frac{1}{r}\frac{d\varphi}{dn}\right)\mathrm{d}\sigma=$$
$$=\oint_{\Sigma}\left(\varphi\frac{\mathrm{d}}{\mathrm{d}n}\frac{1}{r}-\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\right)\mathrm{d}\sigma-\oint_{S}\left(\varphi\frac{\mathrm{d}}{\mathrm{d}n}\frac{1}{r}-\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\right)\mathrm{d}\sigma.$$

On the other hand, on the sphere $S(M,\varrho)$ we have

$$r=\varrho,\ \bar{n}=-\frac{\bar{r}}{r},$$
$$\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)=\bar{n}.\mathrm{grad}\left(\frac{1}{r}\right)=-\frac{\bar{r}}{r}\left(-\frac{1}{r^2}\frac{\bar{r}}{r}\right)=\frac{1}{r^2}|_S=\frac{1}{\varrho}.$$

Therefore, we can write

$$I_L=\frac{1}{\varrho^2}\oint_{\Sigma}\varphi\mathrm{d}\sigma-\frac{1}{\varrho}\oint_{S}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma. \quad (9.2.4)$$

Since $\varphi \in C^2(D)$ we deduce

$$\frac{d\varphi}{dn} \in C^1(D),$$

such that we can use the mean theorem in both sides of Eq. (9.2.4) and deduce that there exists the points $Q_1, Q_2 \in S$ such that

$$I_L = \frac{1}{\varrho^2}\varphi(Q_1)4\pi\varrho - \frac{1}{\varrho}4\pi\varrho\frac{d\varphi}{dn}(Q_2).$$

Here we can pass to the limit as $\varrho \to 0$ and obtain

$$\lim_{\varrho\to 0}\left[4\pi\varphi(Q_1) - 4\pi\varrho\frac{d\varphi}{dn}(Q_2)\right] = 4\pi\varphi(M),$$

because, if $\varrho \to 0$ then the sphere $S(M, \varrho)$ reduces to the point M.

Now, we make some evaluation on the right-hand side of Eq. (9.2.3). Firstly, let us observe that since $\varphi \in C^2(D)$ we deduce that $\Delta\varphi \in C^0(D)$, that is, $\Delta\varphi$ is a continuous function. Also, the distance function r is continuous, such that the function $\Delta\varphi/r$ is continuous. According to the Weierstrass's theorem, a continuous function defined on a closed set is bounded, such that we can write

$$\left|\int_B \frac{1}{r}\Delta\varphi dv\right| \le \int_B \left|\frac{1}{r}\Delta\varphi\right| dv \le M\int_B dv = 4\pi\varrho^3 M.$$

So, we deduce that

$$\lim_{\varrho\to 0}\int_B\left(-\frac{1}{r}\Delta\varphi\right)dv \le \lim_{\varrho\to 0} 4\pi\varrho^3 M = 0.$$

In conclusion, passing to the limit in Eq. (9.2.3), as $\lim\limits_{\varrho\to 0}$, we are led to

$$\lim_{\varrho\to 0}\left[\oint_\Sigma \varphi\frac{d}{dn}\left(\frac{1}{r}\right)d\sigma - \oint_\Sigma \frac{1}{r}\frac{d\varphi}{dn}d\sigma - I_L\right] =$$

$$= \lim_{\varrho\to 0}\left[\int_\Omega -\frac{1}{r}\Delta\varphi dv - \int_{\overline{B}} -\frac{1}{r}\Delta\varphi dv\right].$$

Thus,

$$\oint_\Sigma \varphi\frac{d}{dn}\left(\frac{1}{r}\right)d\sigma - \oint_\Sigma \frac{1}{r}\frac{d\varphi}{dn}d\sigma - 4\pi\varphi(M) = \int_\Omega -\frac{1}{r}\Delta\varphi dv,$$

such that, finally, we have the desired potentials formula

$$\varphi(M)=\frac{1}{4\pi}\oint_\Sigma\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma-\frac{1}{4\pi}\oint_\Sigma\varphi\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma-\frac{1}{4\pi}\int_\Omega\frac{1}{r}\Delta\varphi\mathrm{d}v,$$

and the theorem is proved. ■

An immediate consequence of this theorem is given in the following corollary.

Corollary 9.2.1 *If the scalar function φ is a harmonic function, that is, $\Delta\varphi=0$, then the potentials formula reduces to*

$$\varphi(M)=\frac{1}{4\pi}\oint_\Sigma\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma-\frac{1}{4\pi}\oint_\Sigma\varphi\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma.$$

It is a simple matter to see that this equality is obtained from Eq. (9.2.2) taking into account that $\Delta\varphi$.

This relation says that to obtain the values of the harmonic function φ, in the point from inside of the domain where φ is harmonic, it is sufficient to know the values of the function on the boundary of this domain.

A very important result which follows from the potentials formulas, also in the case of harmonic functions, is proved in the following theorem. This result is called *the Gauss's mean-value formula*.

Theorem 9.2.2 *Let φ be a harmonic function on the domain $D\subset R^3$ and an arbitrary point $M\in D$. Consider the ball $B(M,R)$ such that $B(M,R)\cup S(M,R)=\overline{B}(M,R)\subset D$. Then, the value of function φ in the point M is the mean of the values taken by φ on the sphere $S(M,R)$:*

$$\varphi(M)=\frac{1}{4\pi R^2}\oint_S\varphi(P)\mathrm{d}\sigma.$$

Proof In the particular case when the function φ is harmonic, the potentials formula becomes

$$\varphi(M)=\frac{1}{4\pi}\oint_\Sigma\frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma-\frac{1}{4\pi}\oint_\Sigma\varphi\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma. \tag{9.2.5}$$

But, on the sphere $S(M,R)$ we have

$$r=R,\ \overline{n}=\frac{\overline{n}}{r},$$

$$\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)=\bar{n}.\mathrm{grad}\ \left(\frac{1}{r}\right)=\frac{\bar{r}}{r}\left(-\frac{1}{r^2}\frac{\bar{r}}{r}\right)=-\frac{1}{r^2}|_S=-\frac{1}{R^2}.$$

With the help of this evaluations, Eq. (9.2.5) becomes

$$\varphi(M)=\frac{1}{4\pi R}\oint_\Sigma\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma+\frac{1}{4\pi R^2}\oint_\Sigma\varphi\mathrm{d}\sigma. \tag{9.2.6}$$

Now, we use the second Green's formula for the ball $B(M, R)$ having as boundary the sphere $S(M, R)$ and for the pair of functions φ, which is harmonic, and $\psi \equiv 1$:

$$\oint_{\Sigma}\left(\varphi\frac{\mathrm{d}\psi}{\mathrm{d}n}-\psi\frac{\mathrm{d}\varphi}{\mathrm{d}n}\right)\mathrm{d}\sigma=\int_{B}(\varphi\Delta\psi-\psi\Delta\varphi)\,\mathrm{d}v.$$

Since $\Delta\varphi = 0$, $\Delta\psi = 0$ and $d\psi/dn = 0$ the previous equality leads to

$$\oint_{\Sigma}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}\sigma = 0.$$

Substituting this result in Eq. (9.2.6) we obtain

$$\varphi(M)=\frac{1}{4\pi R^2}\oint_{S}\varphi(P)\mathrm{d}\sigma,$$

that is, the Gauss's formulas and the theorem is concluded. ■

A very useful result is the min-max principle for harmonic functions which is proved in the following theorem.

Theorem 9.2.3 *If φ is a harmonic function on the closed domain $\overline{\Omega} = \Omega \cup \partial\Omega$ and the function φ is not constant, then φ takes its minimum value and the maximum value on the boundary $\partial\Omega$ of the domain Ω.*

Proof Suppose that the function φ is not constant and let us prove that φ takes its maximum value on the boundary $\partial\Omega$ of the domain Ω. The proof is similar to the case of the minimum value. Suppose that there exists a point M_0 inside of the domain Ω such that the maximum value of φ is the value in this point, that is, $\varphi(M_0) > \varphi(M)$, $\forall M$ in a vicinity of M_0. Consider a sphere with M_0 as center which contains all poins for what $\varphi(M_0) > \varphi(M)$. Then

$$\oint_{S}\varphi(M_0)\mathrm{d}\sigma > \oint_{S}\varphi(M)\mathrm{d}\sigma \Rightarrow \varphi(M_0)\oint_{S}\mathrm{d}\sigma > \oint_{S}\varphi(M)\mathrm{d}\sigma.$$

With regards to the last integral we use the Gauss's formula and deduce

$$\varphi(M_0)4\pi R^2 > \varphi(M_0)4\pi R^2 \Rightarrow \varphi(M_0) > \varphi(M_0),$$

which is a contradiction, and the theorem is concluded. ■

Remark. The restriction imposed to the function φ to be not constant is not important, it is necessary only for the method of demonstration. If the function φ is constant then its value in a point is the same both in the case then the point is inside of the domain and in the case when the point is on the boundary. So, we can say that its minimum value and maximum value are taken on the boundary.

9.3 Boundary Values Problems

The problems for the elliptical partial differential equations do not contain the initial data because these equations are models for stationary phenomena. In short, in this context, the problems must determine certain functions which inside of a domain satisfy an elliptical partial differential equation and on the boundary of the domain these functions have a known behavior. In many studies dedicated to the elliptical partial differential equations are well used the following boundary conditions:

(i) the Dirichlet's condition when the value of the function on the boundary is prescribed;
(ii) the Neumann's condition when the value of the normal derivatives of the function on the boundary is prescribed;
(iii) the mixt condition when the value of the function is prescribed on a part of the boundary and on the rest is prescribed the value of the normal derivatives of the function.

As a consequence, there exist three boundary value problem:

(i) the Dirichlet's Problem:

$$\begin{aligned} \Delta u &= f(x, y, z), \ \ (x, y, z) \in \Omega \subset R^3, \\ u|_{\partial\Omega} &= g(x, y, z), \ \ (x, y, z) \in \partial\Omega; \end{aligned} \tag{9.3.1}$$

where the functions f and g are given;
(ii) the Neumann's Problem:

$$\begin{aligned} \Delta u &= f(x, y, z), \ \ (x, y, z) \in \Omega \subset R^3, \\ \frac{\mathrm{d}u}{\mathrm{d}n}|_{\partial\Omega} &= h(x, y, z), \ \ (x, y, z) \in \partial\Omega; \end{aligned} \tag{9.3.2}$$

where the functions f and h are given;
(iii) the mixt Problem:

$$\begin{aligned} \Delta u &= f(x, y, z), \ \ (x, y, z) \in \Omega \subset R^3, \\ u|_{\Sigma_1} &= g(x, y, z), \ \ (x, y, z) \in \Sigma_1; \\ \frac{\mathrm{d}u}{\mathrm{d}n}|_{\Sigma_2} &= h(x, y, z), \ \ (x, y, z) \in \Sigma_2; \end{aligned} \tag{9.3.3}$$

where the functions f and g are given and $\Sigma_1 \cup \Sigma_2 = \partial\Omega$.
Let us, first, study the Dirichlet's problem (9.3.1). Namely, we will prove that the Dirichlet problem has at most one solution.

Theorem 9.3.1 *The solution of the Dirichlet's problem is unique.*

Proof Suppose that the Dirichlet's problem has two solutions, say $u_1(x, y, z)$ and $u_2(x, y, z)$ which correspond to the same right-hand side terms, that is

$$\Delta u_1 = f(x, y, z), \ (x, y, z) \in \Omega$$
$$u_1 = g(x, y, z), \ (x, y, z) \in \partial\Omega,$$

and

$$\Delta u_2 = f(x, y, z), \ (x, y, z) \in \Omega$$
$$u_2 = g(x, y, z), \ (x, y, z) \in \partial\Omega.$$

Denote by $u(x, y, z)$ the difference of these solutions, $u(x, y, z) = u_1(x, y, z) - u_2(x, y, z)$. We intend to prove that

$$u(x, y, z) = 0, \ \forall(x, y, z) \in \Omega.$$

Firstly, we have

$$\Delta u = \Delta u_1 - \Delta u_2 = f - f = 0,$$

such that the function u is harmonic.

On the other hand, for any $(x, y, z) \in \partial\Omega$ we have

$$u(x, y, z) = u_1(x, y, z) - u_2(x, y, z) = g(x, y, z) - g(x, y, z) = 0.$$

Let us suppose there exists a point (x_0, y_0, z_0) such that $u(x_0, y_0, z_0) \neq 0$. Without restricting the generality, we assume that $u(x_0, y_0, z_0) > 0$. As we already proved, u is a harmonic function such that we can apply the min-max principle for harmonic functions and deduce that the maximum value of u is taken on the boundary. But on the boundary the function u takes only the null values. With this contradiction the theorem is concluded. We must say that in the case that $u(x_0, y_0, z_0) < 0$ we use the same min-max principle but with regard to minimum value. If u is a constant function then u takes the same value, say C, in all points of Ω including the boundary. But on the boundary the value of u is zero, therefore $C = 0$. Thus $u = 0$ and then $u_1 = u_2$. ■

With regard to the Neumann's problem we will prove that its solution is not unique. More precisely, we will prove that the difference of any two solutions of the Neumann's problem is a constant.

Theorem 9.3.2 *The solution of the Neumann's problem is determined until an additive constant.*

Proof Let us consider two solutions of the Neumann's problem, that is

$$\Delta u_1 = f(x,y,z),\ (x,y,z)\in\Omega$$
$$\frac{\mathrm{d}u_1}{\mathrm{d}n} = g(x,y,z),\ (x,y,z)\in\partial\Omega,$$

and

$$\Delta u_2 = f(x,y,z),\ (x,y,z)\in\Omega$$
$$\frac{\mathrm{d}u_2}{\mathrm{d}n} = g(x,y,z),\ (x,y,z)\in\partial\Omega.$$

Denote by $u(x,y,z)$ the difference of these solutions, $u(x,y,z) = u_1(x,y,z) - u_2(x,y,z)$. We intend to prove that

$$u(x,y,z) = C = \text{constant},\ \forall(x,y,z)\in\Omega.$$

Firstly, we have

$$\Delta u = \Delta u_1 - \Delta u_2 = f - f = 0,$$

such that the function u is harmonic.

On the other hand, for any $(x,y,z)\in\partial\Omega$ we have

$$\frac{\mathrm{d}u}{\mathrm{d}n} = \frac{\mathrm{d}u_1}{\mathrm{d}n}(x,y,z) - \frac{\mathrm{d}u_2}{\mathrm{d}n}(x,y,z) = g(x,y,z) - g(x,y,z) = 0.$$

Now, remember the first Green's formulas

$$\oint_\Sigma \varphi\frac{\mathrm{d}\psi}{\mathrm{d}n}\mathrm{d}\sigma = \int_\Omega (\text{grad}\ \ \varphi.\text{grad}\ \ \psi + \varphi\Delta\psi)\,\mathrm{d}v.$$

We apply this equality for the pair of functions $\varphi = \psi = u$ and obtain

$$\oint_\Sigma u\frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}\sigma = \int_\Omega \left[(\text{grad}\ \ u)^2 + u\Delta u\right]\mathrm{d}v.$$

But

$$\Delta u = 0,\ \text{in }\Omega,\ \frac{\mathrm{d}u}{\mathrm{d}n} = 0,\ \mathit{on}\ \partial\Omega,$$

such that the previous relation reduces to

$$\int_\Omega (\text{grad}\ \ u)^2\,\mathrm{d}v = 0.$$

So, we have

$$\text{grad}\ \ u = 0 \Rightarrow \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial u}{\partial x} = 0, \; \frac{\partial u}{\partial y} = 0, \; \frac{\partial u}{\partial z} = 0.$$

Thus, the function does not depend on x, y or z, that is u is a constant function and the theorem is concluded. ■

In the following we intend to construct the solutions for the Dirichlet's problem and the Neumann's problem, attached to the Laplace's equation. First, consider the Dirichlet's problem

$$\Delta u = 0, \text{ in } \Omega,$$

$$u = 0, \text{ on } S = \partial\Omega.$$

Taking into account that $\Delta = 0$ and $u|_S = f$ the potentials formula reduces to

$$u(x, y, z) = \frac{1}{4\pi} \oint_S \frac{1}{r} \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}\sigma - \frac{1}{4\pi} \oint_S f \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}n} \left(\frac{1}{r} \right) \mathrm{d}\sigma.$$

So, to know the solution $u(x, y, z)$ we must eliminate, from the above relation, the expression $\mathrm{d}u/\mathrm{d}n$ which is unknown. To this end, we introduce the function $g = g(x, y, z)$ which must satisfy the equations

$$\Delta g = 0, \text{ in } \Omega,$$

$$g = -\frac{1}{r}, \text{ on } S.$$

Now, we apply the second Green's formula for the pair of functions u and g:

$$\oint_S \left(g \frac{\mathrm{d}u}{\mathrm{d}n} - u \frac{\mathrm{d}u}{\mathrm{d}n} \right) \mathrm{d}\sigma = \int_\Omega (g\Delta u - u\Delta g)\, \mathrm{d}v.$$

But $\Delta u = 0$ and $\Delta g = 0$ such the previous relation reduces to

$$\oint_S u \frac{\mathrm{d}g}{\mathrm{d}n} \mathrm{d}\sigma = \oint_S g \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}\sigma = -\oint_S \frac{1}{r} \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}\sigma \Rightarrow$$

$$\Rightarrow \oint_S \frac{1}{r} \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}\sigma = -\oint_S u \frac{\mathrm{d}g}{\mathrm{d}n} \mathrm{d}\sigma = -\oint_S f \frac{\mathrm{d}g}{\mathrm{d}n} \mathrm{d}\sigma.$$

With these evaluations the potentials formula becomes

$$u(x, y, z) = -\frac{1}{4\pi} \oint_S f \frac{\mathrm{d}g}{\mathrm{d}n} \mathrm{d}\sigma - \frac{1}{4\pi} \oint_S f \frac{\mathrm{d}}{\mathrm{d}n} \left(\frac{1}{r} \right) \mathrm{d}\sigma =$$

$$= -\frac{1}{4\pi}\oint_S f\frac{\mathrm{d}}{\mathrm{d}n}\left(g+\frac{1}{r}\right)\mathrm{d}\sigma.$$

Function $g + 1/r$ is called *the Green's function* attached to the domain Ω and it is denoted by G, $G = g + 1/r$. It satisfies the following Dirichlet's problem

$$\Delta G = 0, \text{ in } \Omega$$

$$G|_S = 0, \text{ on } S = \partial\Omega.$$

The Green's function is perfectly determined only by the domain. Therefore, in order to solve the Dirichlet problem on a domain, firstly, we must determine the Green's function for the respective domain and then use the formulas

$$u(x, y, z) = -\frac{1}{4\pi}\oint_S f\frac{\mathrm{d}G}{\mathrm{d}n}\mathrm{d}\sigma.$$

Green's Function for a Sphere

Consider the ball $B(0, R)$ having as boundary the sphere $S(0, R)$. Let M_0 be an arbitrary fixed point in the ball and another point M_1 such that

$$OM_0.OM_1 = R^2.$$

On the sphere we take the point M such that the segment M_1M is tangent to the sphere. Therefore, we have

$$\triangle OM_0M\,\tilde{\triangle}\, OM_1M$$

and can write

$$\frac{OM_0}{OM} = \frac{OM}{OM_1} = \frac{M_0M}{M_1M}. \tag{9.3.4}$$

Using the notations

$$OM_0 = d,\ OM_1 = d_1,\ OM = R,\ \overline{M}_0M = \overline{r},\ \overline{M}_1M = \overline{r}_1,$$

the qualities (9.3.4) become

$$\frac{d}{R} = \frac{R}{d_1} = \frac{r}{r_1}. \tag{9.3.5}$$

Define function g by

$$g = -\frac{R}{d}\frac{1}{r_1},$$

or

$$g(M) = -\frac{OM}{d}\frac{1}{|M_1 M|}.$$

It is easy to see that for $M \in S(0, R)$ we have

$$g(M) = -\frac{1}{r}.$$

Moreover, by direct calculations, it is easy to prove that the function g is harmonic. Therefore, the Green's function for the sphere is

$$G = g + \frac{1}{r} = \frac{1}{r} - \frac{R}{d}\frac{1}{r_1}.$$

Now, because we know the Green's function of the sphere, we can find the solution of the Dirichlet's problem on the ball, namely

$$u(x, y, z) = -\frac{1}{4\pi}\oint_S f \frac{\mathrm{d}G}{\mathrm{d}n}\mathrm{d}\sigma. \tag{9.3.6}$$

This solution will be completely determinate if we find dG/dn. Thus

$$\frac{\mathrm{d}G}{\mathrm{d}n} = \overline{n}.\text{grad } G = \overline{n}\left(\frac{1}{r} - \frac{R}{d}\frac{1}{r_1}\right) =$$

$$= \overline{n}\left(\text{grad } \frac{1}{r} - \text{grad } \frac{R}{dr_1}\right) = \overline{n}\left[-\frac{1}{r^2}\frac{\overline{r}}{r} - \frac{R}{d}\left(-\frac{1}{r_1^2}\frac{\overline{r}_1}{r_1}\right)\right] =$$

$$= \frac{\overline{nr}}{r^3} + \frac{R}{d}\frac{\overline{nr}_1}{r_1^3}.$$

But, in our case

$$\overline{r} = \overline{M}_0 M = \overline{O}M - \overline{O}M_0 \Rightarrow \overline{nr} = \overline{n}\left(\overline{O}M - \overline{O}M_0\right) = R - d\cos\theta,$$

$$\overline{r}_1 = \overline{M}_1 M = \overline{O}M - \overline{O}M_1 \Rightarrow \overline{nr}_1 = \overline{n}\left(\overline{O}M - \overline{O}M_1\right) = R - d_1\cos\theta,$$

where θ is the angle $\widehat{MOM_0}$. With these calculations the derivative of the function G becomes

$$\frac{dG}{dn} = \frac{d\cos\theta - R}{r^3} + \frac{R}{d}\frac{R - d_1\cos\theta}{r_1^3.}$$

From Eq. (9.3.5) we obtain

$$r_1 = \frac{rR}{d},\ d_1 = \frac{R^2}{d},$$

such that

$$\frac{dG}{dn} = \frac{d\cos\theta - R}{r^3} + \frac{R}{d}\frac{R - R^2/d\cos\theta}{1}\frac{d^3}{r^3R^3} =$$

$$= \frac{1}{r^3}\left[d\cos\theta - R + \frac{1}{d^2}(d - R\cos\theta)\frac{d^3}{R}\right] = \frac{1}{r^3}\left[d\cos\theta - R + \frac{d}{R}(d - R\cos\theta)\right] =$$

$$= \frac{1}{r^3R}\left(d\cos\theta - R^2 + d^2 - dR\cos\theta\right) = \frac{d^2 - R^2}{r^3R}.$$

Finally, we introduce the derivative of the Green's function in Eq. (9.3.6) and the solution of the Dirichlet's problem is completely determined

$$u(x, y, z) = \frac{r^2 - d^2}{4\pi R}\oint_S f\frac{f(x, y, z)}{r^3}\mathrm{d}\sigma.$$

Dirichlet's Problem for a Circle

In this paragraph we give a complete solution for the Dirichlet's problem attached to a circle centered in the origin and having the radius R. With other words, we find a function $u = u(x, y)$ which is harmonic inside of a circle and has known values on the circumference of the circle:

$$\Delta u(x, y) = 0,\ x^2 + y^2 < R^2,$$

$$u(x, y) = f(x, y),\ x^2 + y^2 = R^2,$$

where $f = f(x, y)$ is a given function.

First, we write the Laplacean in polar coordinates:

$$x = r\cos\theta,\ y = r\sin\theta,\ \text{where } r = \sqrt{x^2 + y^2},\ \theta = \text{arctg}\frac{y}{x}.$$

The first derivatives of the function u are

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial \theta}\frac{x}{r^2}.$$

Then, the second derivatives of the function u become

$$\frac{\partial^2 u}{\partial x^2} = \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial \theta^2} + \frac{r^2 - x^2}{r^3}\frac{\partial u}{\partial r} + \frac{2xy}{r^4}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial \theta^2} + \frac{r^2 - y^2}{r^3}\frac{\partial u}{\partial r} - \frac{2xy}{r^4}\frac{\partial u}{\partial \theta}.$$

The Laplacean in polar coordinates becomes

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r},$$

such that we find the following form for the Laplace's equation:

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Consequently, the Dirichlet's problem can be stated in the form

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0,\ r < R,$$

$$u(r, \theta) = f(\theta),\ r = R.$$

To find the solution of this problem we use the separation of variables method. Thus, we find the solution in the form

$$u(r, \theta) = V(r).W(\theta).$$

So, for the derivative we find the expressions

$$\frac{\partial u}{\partial r} = V'.W,\ \frac{\partial^2 u}{\partial r^2}V''.W,$$

$$\frac{\partial u}{\partial \theta} = V.W',\ \frac{\partial^2 u}{\partial \theta^2}V.W'',$$

such that the Laplace's equation becomes

$$r^2V''.W + rV'.W + V.W'' = 0.$$

From here, dividing by the product $V.W$, we obtain

$$r^2\frac{V''}{V} + r\frac{V'}{V} + \frac{W''}{W} = 0 \Rightarrow r^2\frac{V''}{V} + r\frac{V'}{V} = -\frac{W''}{W} = k,\ k = \text{constant}.$$

Therefore, from the Laplace's equation we find two ordinary differential equations

$$W'' - kW = 0,$$

$$r^2 \frac{V''}{V} + r\frac{V'}{V} = -k.$$

We study the first equation. Supposing that $k > 0$ we find the solution

$$W(\theta) = C_1 e^{-\sqrt{k}\theta} + C_2 e^{\sqrt{k}\theta},$$

which is not appropriate because it is not periodical. If $k = 0$, we find

$$W'' = 0 \Rightarrow W(\theta) = C_1\theta + C_2,$$

which also, is not appropriate because it is not periodical.

In the case $k < 0$ we use the notation $k = -\lambda^2$ and then

$$W'' + \lambda^2 W = 0 \Rightarrow W(\theta) = C_1 \cos \lambda\theta + C_2 \sin \lambda\theta.$$

This solution is periodical if

$$W(\theta + 2\pi) = C_1 \cos \lambda(\theta + 2\pi) + C_2 \sin \lambda(\theta + 2\pi) = C_1 \cos(\lambda\theta + 2n\pi) + C_2 \sin(\lambda\theta + 2n\pi),$$

such that $\lambda = n, \ n = 1, 2, ...$

So, we find an infinite number of particular solutions of the form

$$W_n(\theta) = A_n \cos n\theta + B_n \sin n\theta. \tag{9.3.7}$$

Let us study the equation for the function V:

$$r^2 \frac{V''}{V} + r\frac{V'}{V} + k = 0,$$

which, with the above value of k becomes

$$r^2 V'' + rV' - n^2 V = 0.$$

This is a differential equation of second order of Euler type for the function V depending on the variable r. As usual, we make the change of variable $r = e^t$ and then

$$dr = e^t \mathrm{d}t \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = e^t, \ \frac{\mathrm{d}t}{\mathrm{d}r} = e^{-t}.$$

Consequently,

$$V' = \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\mathrm{d}v}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = \dot{V} e^{-t},$$

$$V'' = \frac{dv'}{dr} = \frac{dv'}{dt}e^{-t} = e^{-t}\left(-e^{-t}\dot{V} + e^{-t}\ddot{V}\right) = e^{-2t}\left(\ddot{V} - \dot{V}\right).$$

Then, the equation of function V becomes

$$e^{2t}\left(\ddot{V} - \dot{V}\right)e^{-2t} + e^{t}\dot{V}e^{-t} - n^2 V = 0 \Rightarrow \ddot{V} - n^2 V = 0,$$

having the solution

$$V(t) = D_n e^{-nt} + C_n e^{nt} = D_n\left(e^t\right)^{-n} + C_n\left(e^t\right)^n \Rightarrow$$

$$\Rightarrow V(r) = D_n r^{-n} + C_n r^n.$$

But the function r^{-n} does not exist in the origin such that we must put $D_n = 0$, therefore the particular solutions are

$$V_n = C_n r^n,\ n = 1, 2, ...$$

Consequently, taking into account the solution (9.3.7) for the equation of W, the Laplace's equation has the particular solutions

$$u_n(r, \theta) = W_n(\theta).V_n(r) = C_n r^n\left(A_n \cos n\theta + B_n \sin n\theta\right),$$

or, with the convention $C_n.A_n \to A_n,\ C_n.B_n \to B_n$,

$$u_n(r, \theta) = \left(A_n \cos n\theta + B_n \sin n\theta\right) r^n.$$

A linear combination of these particular solutions gives the general solution

$$u(r, \theta) = \sum_0^\infty \left(A_n \cos n\theta + B_n \sin n\theta\right) r^n,$$

or

$$u(r, \theta) = \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n,$$

where $a_0 = 2A_0,\ a_n = A_n,\ b_n = B_n,\ n \geq 1.$
The Dirichlet's condition leads to

$$u(R, \theta) = \frac{a_0}{2} + \sum_1^\infty \left(R^n a_n \cos n\theta + R^n b_n \sin n\theta\right) = f(\theta).$$

The Fourier coefficients of this series are

$$a_0 = \frac{2}{T}\int_0^T f(t)\mathrm{d}t = \frac{1}{\pi}\int_0^T f(t)\mathrm{d}t$$

$$R^n a_n = \frac{1}{\pi}\int_0^T f(t)\cos nt\mathrm{d}t,\ R^n b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\mathrm{d}t,$$

such that the solution becomes

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t + \sum_1^\infty \left[\frac{1}{\pi R^n}\cos n\theta \int_0^{2\pi} f(t)\cos nt\mathrm{d}t + \right.$$

$$\left. + \frac{1}{\pi R^n}\sin n\theta \int_0^{2\pi} f(t)\sin nt\mathrm{d}t\right] r^n.$$

We can write the solution in the following forms

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t + \sum_1^\infty \frac{1}{\pi}\left(\frac{r}{R}\right)^n \int_0^{2\pi} f(t)\cos n(\theta - t)\mathrm{d}t =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t + \frac{1}{\pi}\sum_1^\infty \int_0^{2\pi} f(t)\left(\frac{r}{R}\right)^n \cos n(\theta - t)\mathrm{d}t =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t + \frac{1}{\pi}\int_0^{2\pi} f(t)\sum_1^\infty \left(\frac{r}{R}\right)^n \cos n(\theta - t)\mathrm{d}t.$$

Finally, we can write

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(t)\left[\frac{1}{2} + \sum_1^\infty \left(\frac{r}{R}\right)^n \cos n(\theta - t)\right]\mathrm{d}t. \tag{9.3.8}$$

Now, we make some evaluations on the series

$$\sum_1^\infty \left(\frac{r}{R}\right)^n \cos n(\theta - t),$$

by means the series

$$\sum_1^\infty \left(\frac{r}{R}\right)^n e^{in(\theta - t)} = \sum_1^\infty \left(\frac{r}{R}e^{i(\theta - t)}\right)^n,$$

which is convergent because

$$|q| = \left|\left(\frac{r}{R}\right) e^{i(\theta-t)}\right| = \frac{r}{R} < 1.$$

As it is well known the sum of this series is $q/(1-q)$, that is

$$\sum_{1}^{\infty} \left(\frac{r}{R} e^{i(\theta-t)}\right)^n = \frac{q}{1-q}.$$

Let us evaluate the sum of the series:

$$\frac{q}{1-q} = \frac{re^{i(\theta-t)}}{R - re^{i(\theta-t)}} =$$

$$= \frac{r[\cos(\theta-t) + i\sin(\theta-t)]}{R - \cos(\theta-t) - ir\sin(\theta-t)} =$$

$$= \frac{r[\cos(\theta-t) + i\sin(\theta-t)][R - \cos(\theta-t) + ir\sin(\theta-t)]}{[R - r\cos(\theta-t)]^2 + r^2\sin^2(\theta-t)} =$$

$$= \frac{r[R\cos(\theta-t) - r + iR\sin(\theta-t)]}{R^2 - 2rR\cos(\theta-t) + r^2}.$$

Then our series becomes

$$\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos n(\theta-t) = Re \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n e^{in(\theta-t)} =$$

$$= \frac{r[R\cos(\theta-t) - r]}{R^2 - 2rR\cos(\theta-t) + r^2}.$$

For the solution we obtain the following expressions

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(t)\left[\frac{1}{2} + \frac{r[R\cos(\theta-t) - r]}{R^2 - 2rR\cos(\theta-t) + r^2}\right]\mathrm{d}t =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-t) + r^2}\mathrm{d}t.$$

Finally, we can write the following form of the solution

$$u(r,\theta) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(t)}{R^2 - 2rR\cos(\theta-t) + r^2}\mathrm{d}t,$$

which is called *the Poisson's solution* for the Dirichlet's problem.

Now, we apply the Poisson's form of the solution to obtain the solution for a useful application.

Application. Find the distribution of the electrical field in a plane disk knowing that it is 1 on the superior circumference and 0 on the inferior circumference.
Solution. We apply the Poisson's formula for the function

$$f(t)=\begin{cases}1, & t\in[0.\pi]\\ 0, & t\in(\pi.2\pi].\end{cases}$$

Thus, the above relation becomes

$$u(r,\theta)=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{1}{R^2-2rR\cos(\theta-t)+r^2}\mathrm{d}t.$$

To compute the integral we make the change of variable:

$$\mathrm{tg}\frac{t-\theta}{2}=\tau\Rightarrow t-\theta=2\mathrm{arctg}\,\tau\Rightarrow \mathrm{d}t=\frac{2}{1+t^2}\mathrm{d}\tau.$$

Also, because $t\in[0,\pi]$, we deduce

$$\tau\in\left[-\mathrm{tg}\frac{\theta}{2},\,\mathrm{ctg}\frac{\theta}{2}\right].$$

The integrant becomes

$$R^2-2rR\frac{1-\tau^2}{1+\tau^2}+r^2=\frac{(R-r)^2+\tau^2(R+r)^2}{1+\tau^2},$$

such that the solution receives the forms

$$u(r,\theta)=\frac{R^2-r^2}{2\pi}\int_{-tg\,\theta/2}^{ctg\,\theta/2}\frac{1+\tau^2}{(R-r)^2\left[1+\left(\frac{R+r}{R-r}\tau\right)^2\right]}\frac{2}{1+\tau^2}\mathrm{d}\tau=$$

$$=\frac{1}{\pi}\int_{-tg\,\theta/2}^{ctg\,\theta/2}\frac{R+r}{R-r}\frac{1}{1+\left(\frac{R+r}{R-r}\tau\right)^2}\mathrm{d}\tau=\frac{1}{\pi}\mathrm{arctg}\,\frac{R+r}{R-r}\tau\Big|_{-tg\,\theta/2}^{ctg\,\theta/2}=$$

$$=\frac{1}{\pi}\left[\mathrm{arctg}\,\frac{R+r}{R-r}\,ctg\,\theta/2+\mathrm{arctg}\,\frac{R+r}{R-r}\,tg\,\theta/2\right].$$

Finally, we find the solution

$$u(r,\theta)=\frac{1}{\pi}\mathrm{arctg}\,\frac{r^2-R^2}{2rR\sin\theta}.$$

Neumann's Problem

In the following we intend to construct the solution for the Neumann's problem defined by the relations

$$\Delta u = 0, \text{ in } \Omega,$$
$$\frac{du}{dn} = h, \text{ on } S = \partial\Omega.$$

Taking into account that $\Delta = 0$ and $u|_S = f$ the potentials formula reduces to

$$u(x, y, z) = \frac{1}{4\pi}\oint_S h\frac{1}{r}d\sigma - \frac{1}{4\pi}\oint_S u\frac{d}{dn}\left(\frac{1}{r}\right)d\sigma. \tag{9.3.9}$$

So, to know the solution $u(x, y, z)$ we must eliminate, from the above relation (9.3.9), the expression u which is unknown. Let us write the second Green's formula

$$\oint_S \left(\varphi\frac{d\psi}{dn} - \psi\frac{d\varphi}{dn}\right)d\sigma = \int_\Omega (\varphi\Delta\psi - \psi\Delta\varphi)dv,$$

for the pair of functions $\varphi = h$ and $\psi = u$:

$$\oint_S \left(u\frac{dh}{dn} - h\frac{du}{dn}\right)d\sigma = 0 \Rightarrow \oint_S h\frac{du}{dn}d\sigma = \oint_S u\frac{dh}{dn}d\sigma.$$

Now, consider the function g such that $\Delta g = 0$ in Ω. In the same manner as above we obtain

$$\oint_S \left(g\frac{du}{dn} - u\frac{dg}{dn}\right)d\sigma = 0 \Rightarrow \frac{1}{4\pi}\oint_S \left(g\frac{du}{dn} - u\frac{dg}{dn}\right)d\sigma = 0.$$

If we add this equality, term by term, to the equality (9.3.9) we are led to

$$u(x, y, z) = \frac{1}{4\pi}\oint_S \left[h\frac{du}{dn} - u\frac{dg}{dn} + \frac{1}{r}\frac{du}{dn} - u\frac{d}{dn}\left(\frac{1}{r}\right)\right]d\sigma =$$
$$= \frac{1}{4\pi}\oint_S \left[\left(g + \frac{1}{r}\right) - u\frac{d}{dn}\left(g + \frac{1}{r}\right)\right]d\sigma.$$

Function $g + 1/r$ is called *the Green's function* attached to the domain Ω and it is denoted by G, $G = g + 1/r$. If we impose to the function G to satisfy the condition

$$\frac{dG}{dn} = \frac{4\pi}{A},$$

where A is the area of the surface S, then the solution receives the form

$$u(x,y,z) = \frac{1}{4\pi}\oint_S G\frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}\sigma - \frac{1}{4\pi}\oint_S \frac{4\pi}{A}u\mathrm{d}\sigma =$$

$$= \frac{1}{4\pi}\oint_S G\frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}\sigma - \frac{1}{A}\oint_S u\mathrm{d}\sigma.$$

The last integral does not depend on the point $M = (x, y, z)$, that is, the integral is a constant:

$$-\frac{1}{A}\oint_S u\mathrm{d}\sigma = C = \text{constant}.$$

Taking into account that

$$\frac{\mathrm{d}u}{\mathrm{d}n}|_S = h,$$

the solution of the Neumann's problem is

$$u(x,y,z) = \frac{1}{4\pi}\oint_S hG\mathrm{d}\sigma + C,$$

G being the Green's function of the domain Ω.

Therefore, to determine the function u we must, firstly, determine the Green's function of the domain and this, in fact, means the determination of the function g which is harmonic in the domain and on the surface S satisfies the condition

$$\frac{\mathrm{d}g}{\mathrm{d}n} = \frac{4\pi}{A} - \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right).$$

Remark. The Neumann's problem does not have always a solution. To find the condition when the Neumann's problem has a solution, we use the first Green's formulas

$$\oint_S \varphi\frac{\mathrm{d}\psi}{\mathrm{d}n}\mathrm{d}\sigma = \int_\Omega (\text{grad } \ \varphi.\text{grad } \ \psi + \varphi\Delta\psi)\,\mathrm{d}v$$

for the pair of functions $\varphi = 1$ and $\psi = u$:

$$\oint_S \frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}\sigma = \int_\Omega \Delta u\mathrm{d}v = 0 \Rightarrow \oint_S \frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}\sigma = 0 \Rightarrow \oint_S h\mathrm{d}\sigma = 0.$$

Also, the Neumann's problem does not have always a solution even in the more general case when the Laplace's equation is replaced by the Poisson equation, that is for the problem

$$\Delta u = f \text{ in } \Omega$$

$$\frac{\mathrm{d}u}{\mathrm{d}n} = h \text{ on } S = \partial\Omega.$$

Indeed, we use the first Green's formulas for the pair of functions $\varphi = 1$ and $\psi = u$:

$$\oint_S \frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}\sigma = \int_\Omega \Delta u \mathrm{d}v.$$

Therefore, the Neumann's problem attached to the Poisson equation, admits a solution if

$$\oint_S h\mathrm{d}\sigma = \int_\Omega f\mathrm{d}v.$$

At the final part of this paragraph we indicate a procedure to solve the Neumann's problem in the particular case when the domain Ω is a sphere centered in the origin and having the radius R. Let us use the spherical coordinates

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta, \end{cases}$$

where $r \in [0, R]$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$.

We consider only the particular case when the data on the boundary depends only on the angle θ, that is

$$\frac{\mathrm{d}u}{\mathrm{d}n}|_{x^2+y^2+z^2=R^2} = h(\theta).$$

We find the solution in the form $u = u(r, \theta)$, such that we must write the Laplace's equation in polar coordinates

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0.$$

Now, we can use the method of separation of variables, that is we find the solution in the form $u(r, \theta) = R(r)T(\theta)$. The derivatives of the function u become

$$\frac{\partial u}{\partial r} = R'T,\ \frac{\partial}{\partial\theta} = RT' \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2R'T\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta RT'\right) = 0.$$

The Laplace's equation becomes

$$2rR'T + r^2R''T + \frac{1}{\sin\theta}\left(RT'\cos\theta + RT''\sin\theta\right) = 0,$$

or, equivalently,

$$2r\frac{R'}{R} + r^2\frac{R''}{R} + \frac{1}{\sin\theta}\left(\frac{T'}{T}\cos\theta + \frac{T''}{T}\sin\theta\right) = 0.$$

So, we obtain two ordinary differential equations

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} = k,$$

$$\frac{1}{\sin\theta}\left(\frac{T'}{T}\cos\theta + \frac{T''}{T}\sin\theta\right) = -k$$

where k is a constant.

These equations can be written in the form

$$r^2R'' + 2rR' - kR = 0,$$

$$T''\sin\theta + T'\cos\theta + kT\sin\theta = 0,$$

and the readers can, easy, find their solutions.

Chapter 10
Optimal Control

10.1 Preparatory Notions

In this section we will introduce some notions and results which are specific to the functional analysis and are necessary in the whole present chapter. Since we consider these notions and results being subordinate to the main objectives of this chapter, we shall renounce to prove the results. Our readers are invited, for more details, to consult the titles cited in the bibliography dedicated to functional analysis and convex analysis.

Let us denote by X a real Banach space and by X^* its dual, that is the set of all linear and continuous functionals defined on the space X. By convention, we will denote by (x^*, x) the value of the functional x^* in the point x. For an arbitrary fixed functional $x^* \in X^*$, we define its *seminorm* p_{x^*} by

$$p_{x^*}(x) = x^*(x) = (x^*, x),$$

such that we get the following family of seminorms:

$$\{p_{x^*}\}_{x^* \in X^*}.$$

With the aid of this family of seminorms we can introduce a new topology on the Banach space X, called the *weak topology*, to make it distinctly of the initial topology of the space X, called the *strong topology*. In the weak topology the convergence is defined by

$$x_n \rightharpoonup x \Leftrightarrow (x_n^*, x) \to (x^*, x), \ \forall x^* \in X^*,$$

where we have used the notation $\rightharpoonup$ to designate the weak convergence and $\to$ to designate the strong convergence.

Similarly, for an arbitrarily fixed element $x \in X$, we define *the seminorm* p_x by

$$p_x(x^*) = x^*(x) = (x^*, x),$$

M. Marin and A. Öchsner, *Complements of Higher Mathematics*,
https://doi.org/10.1007/978-3-319-74684-5_10

such that we have the family of seminorms $\{p_x\}_{x\in X}$. With the help of this family of seminorms we can introduce on the Banach space X^* a new topology, and, also, we will called it *the weak topology*, too. In this weak topology the convergence is defined by

$$x_n^* \rightharpoonup x^* \Leftrightarrow (x_n^*, x) \to (x^*, x), \ \forall x \in X,$$

Let us denote by X^{**} the dual space of the space X^*. As it is well known, we always have the inclusion $X \subset X^{**}$, since the space$X$ can be sunk in X^{**} through the application

$$x \to f_x, \ f_x \in X^{**}, \ f_x(x^*) = x^*(x), \ \forall x^* \in X^*.$$

If we have the inclusion $X^{**} \subset X$, too (and thus, according to the previous statement, we have $X = X^{**}$) then we say that X is a *reflexive space*. A known example of reflexive space is offered by any Hilbert space. Also, the space $L^p(\Omega)$ for $p > 1$ is a reflexive space, but the space $L^1(\Omega)$ is not a reflexive space. In all what follows we will frequently use the notation $\bar{R}$ to designate the space of all real numbers completed by ∞, that is

$$\bar{R} = R \cup \{\infty\}.$$

The function $\varphi : X \to \bar{R}$ is called a *convex function* if it is subadditive with respect to any convex combination of elements from the Banach space X, that is

$$\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y), \ \forall x, y \in X, \ \forall \lambda \in [0, 1].$$

We say that the function $\varphi : X \to \bar{R}$ is *inferior semi-continuous* (and we will abbreviate it by i. s. c.), if

$$\liminf_{y\to x} \varphi(y) \ge \varphi(x),$$

where, by definition, we have that

$$\liminf_{y\to x} \varphi(y) = \sup_{V\in\mathcal{V}(x)} \inf_{y\in V} \varphi(x).$$

From this definition we deduce that we always have

$$\varphi(x) \ge \liminf_{y\to x} \varphi(y)$$

and then we can say that the function φ is i. s. c. if

$$\liminf_{y\to x} \varphi(y) = \varphi(x).$$

An equivalent formulation of the fact that the function φ is inferior semi-continuous: the bounded set $\{x \ : \ \varphi \leq \lambda\}$ is closed.

We remember that *the effective domain* of a function φ is the set denoted by $D(\varphi)$ and defined by

$$D(\varphi) = \{x \in X : \varphi(x) < \infty\} \subset X.$$

Theorem 10.1.1 *If X is a Banach space and the function $\varphi : X \to \bar{R}$ is convex and i. s. c. , then it is inferior bounded by an affine function, that is $\exists \alpha \in R$ and $x_0^* \in X^*$ such that*

$$\varphi(x) \geq (x_0^*, x) + \alpha, \ \forall x \in X.$$

Theorem 10.1.2 *Let X be a Banach space and the function $\varphi : X \to \bar{R}$ which is a convex function, i. s. c. and satisfies the condition:*

$$\lim_{\|x\| \to \infty} \varphi(x) = \infty,$$

where $\|.\|$ represents the strong norm of the space X.

Then, the function φ actually attains its minimum on the space X.

We remeber that the function $\varphi : X \to R$ is a *Gateaux differentiable* function in the point x_0 if:

- there exists the gradient of φ in x_0, denoted by grad $\varphi(x_0) \in X^*$, and:

$$\lim_{\lambda \to 0} \frac{\varphi(x_0 + \lambda x) - \varphi(x_0)}{\lambda} = (\text{grad } \varphi(x_0), x) \, , \ \forall x \in X.$$

We say that $\varphi : X \to R$ is a *Frechet differentiable* function in the point x_0 if

- there exists the gradient of φ in x_0, grad $\varphi(x_0) \in X^*$ and:

$$\varphi(x_0 + x) - \varphi(x_0) = (\text{grad } \varphi(x_0), x) + \omega(x), \ \forall x \in X,$$

where the function $\omega(x)$ has the property:

$$\lim_{\|x\| \to 0} \omega(x) = 0.$$

If X is a Banach space and the function $\varphi : X \to \bar{R}$ is a convex and i. s. c. function, then *the subgradient* of the function φ in the point x_0, denoted by $\partial\varphi(x_0)$, is defined by

$$\partial\varphi(x_0) = \left\{x^* \in X^* : \varphi(x_0) - \varphi(x) \leq \left(x^*, x_0 - x\right), \ \forall x \in X\right\}.$$

We say that the function φ is *subdifferentiable* in the point x_0 if the subgradient of the function φ computed in x_0 is non-empty, that is $\partial\varphi(x_0) \neq \emptyset$.

From the definition, it is easy to see that the subdifferential of a function is a multivocal mapping, more exactly, is a multivocal operator.

In the case of a Banach space X, an example of subdifferential is given by application of the duality F of the space X, defined by:

$$F(x) = \left\{x^* \in X^* : \left(x^*, x\right) = \|x\|_X^2 = \|x^*\|_{X^*}^2\right\}.$$

It is easy to prove that for the function $\varphi : X \to R$, defined by

$$\varphi(x) = \frac{1}{2}\|x\|_X^2,$$

its subdifferential is just the application of duality, that is $\partial\varphi(x_0) = F(x_0)$.

Let X and Y be two Banach spaces and A a multivocal operator, $A : X \to Y$. We say that A is a *monotone operator* if:

$$(y_1 - y_2, x_1 - 2_1) \geq 0,\ \forall x_1, x_2 \in X,\ \ y_1 \in Ax_1,\ \ y_2 \in Ax_2.$$

In the particular case when the operator A is univocal, the definition of the monotony received the following simplified form:

$$(Ax_1 - Ax_2, x_1 - 2_1) \geq 0,\ \forall x_1, x_2 \in X.$$

If the monotone operator A does not admit a proper extension which is also, a monotone operator, then we say that A is a *maximal monotone operator*.

An useful example of multivocal operator which is maximal monotone is offered by the subdifferential of a convex and i. s. c. function. With the help of the subdifferential we can characterize the minimum of a function. More exactly, the function $\varphi : X \to \bar{R}$ which is convex and i. s. c. . attains its minimum, that is, $\exists x_0 \in X$ such that

$$\inf_{x \in X} \varphi(x) = \varphi(x_0),$$

if and only if $0 \in \partial\varphi(x_0)$, where 0 is the null element of the Banach space X.

To make easy the passing from a problem of minimum to a problem of maximum for a given function it was introduced the notion of the *conjugated function* attached of the respective function. If the function $\varphi : X \to \bar{R}$ is convex and i. s. c. then its conjugated function, denoted by φ^*, is defined by

$$\varphi^* : X^* \to R,\ \varphi^*(x^*) = \sup_{x \in X}\left\{\left(x^*, x\right) - \varphi(x)\right\}.$$

The main properties of the conjugated function are:

- 1. $\varphi^*(x^*) + \varphi(x) \geq (x^*, x)\,,\ \ \forall x \in X,\ \ x^* \in X^*$, the equality takes place if and only if $x^* \in \partial\varphi(x)$.
- 2. $\varphi^{**}(x) \leq \varphi(x),\ \forall x \in X,\ x^* \in X^*$, the equality holds if and only if $x^* \in \partial\varphi(x)$.
- 3. $\partial\varphi^* = (\partial\varphi)^{-1}$.

The passing from a problem of minimum to one of maximum and conversely, can be made based on the following theorem, due to Fenchel.

Theorem 10.1.3 *If the functions* $\varphi, \psi : X \to \bar{R}$ *are convex and i. s. c. and one from the two following conditions holds:*

$$[Int D(\varphi)] \cap D(\psi) \neq \emptyset \ or\ [Int D(\psi)] \cap Int D(\varphi) \neq \emptyset,$$

then

$$\inf_{x \in X} \{\varphi(x) + \psi(x)\} = \sup_{x^* \in X^*} \left\{-\varphi^*(x^*) - \psi^*(-x^*)\right\}.$$

10.2 Problems of Optimal Control

Suppose that a physical system lies, at a given moment, in the position x_0. From this position it can continue its evolution on different trajectories. If we impose to the system some conditions which determine it to displace on such a trajectory which is more suitable with regard to the proposed purpose, we say that we apply to the system a *optimal command*. Therefore, it is possible to interced towards the system by an external intervention, a command or an *optimal control* which is imposed to the system to execute the desired movement. For instance, the system can be ordered or supervised such that its evolution takes place in an optimal time or by a minimal consumption of energy.

The mathematical formulation of these problems, in the classical manner, is the object of the study of the chapter *Variational Calculus* which has been exposed already in a previous chapter (Chap. 5).

In the last period of time, in view of the formulation and solution of these problems has been appeared a new theory, called the *Theory of Optimal Control*. In all what follows we intend to expose just the basic notions and results of this theory. Otherwise, the theory of optimal control is a very abundant and pretentious theory and, therefore, it is only accessible to a small number of readers.

Essentially, this theory is based on *the maximum principle*, also called, the Pontreagin's principle, because this great Russian mathematician is the author of this principle.

Let us consider a mobil system which can be commanded (or controlled) such that it evolves according to the desired purpose. Usually, the movement of the system is mathematical modeled by a system of differential equations of the form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i\left(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m\right),\ \ i = \overline{1, n}, \tag{10.2.1}$$

in which the quantities x_i characterize the position of the mobile and the vector field $x = (x_1, x_2, ..., x_n)$ is called *the state variable*. Also, in formula (2.1) u_i are the parameters of the external action which act on the evolution of the system, and the vector field $u = (u_1, u_2, ..., u_m)$ is called *variable of command* or *variable of control*.

Let us consider the functional:

$$J = \int_0^T F\,(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m)\,\mathrm{d}t. \tag{10.2.2}$$

If for $t \in [0, T]$ are given the parameters of control:

$$u_j = u_j(t),\;\; j = \overline{1, m} \tag{10.2.3}$$

and the initial conditions:

$$x_i(0) = x_i^0,\;\; i = \overline{1, n}, \tag{10.2.4}$$

then the system (10.2.1) possesses only one solution and the integral (10.2.2) has only one value, which will be the optimal value with respect to the desired purpose.

Suppose that there exists a control u given as in Eq. (10.2.3) which determines the system, placed in the initial state (10.2.4) to arrive at the final state which we imposed to it:

$$x_i(T) = x_i^1,\;\; i = \overline{1, n}. \tag{10.2.5}$$

The system will move on a certain trajectory starting from the position (x_i^0) to the position (x_i^1) and, as a consequence, the functional J from Eq. (10.2.2) takes a certain value. Corresponding to another imposed command to the system, u, it will evolve between the two positions on the other trajectory. As a consequence, the functional J, computed through the new trajectory, will take another value. Thus, it appears the problem of finding that command (or that control) which determines the evolution of the system from the initial state to the final state through a trajectory on which the functional J attains its minimum value.

From a physical point of view, the parameters of control can represent, for instance: the quantity of fuel delivered to an engine, temperature, the intensity of an electric current, and so on.

We must specify the fact that in running practical problems the parameters of control do not take any arbitrary values but these parameters are subjected to certain restrictions. For instance, in the case of a plane command, if the parameters u_1 and u_2 are the components of a sub-unit vector field, then the respective command must carry out the following condition:

$$u_1^2 + u_2^2 \leq 1.$$

In the general case, it is assumed that $(u_1, u_2, ..., u_m) \in U$, where U is a set from a $m-$dimensional space and is called *the domain of command* or *the domain of control*. It is possible that, at different moments, the domain of command is different, that is $U = U(t)$.

Also, in the concrete problems of optimal control there are imposed some restrictions on the state, called *restrictions of phase*, that is, the state vector field cannot go out of a certain region of the $n-$dimensional space R^n, that will be written in the form:

$$(x_1, x_2, ..., x_n) \in G \subset R^n.$$

Also, in this case we have $G = G(t),\ t \in [0, T]$.

More general, it is possible to appear some mixt restrictions which are imposed, simultaneous, on the command, and, also, on the state:

$$(x(t), u(t)) \in D = D(t) \subset R^n \times R^m,\ t \in [0, T].$$

If we can write $D(t) = G(t) \times U(t)$ we say that the restrictions on the state and on the command are separately imposed.

A very important particular case is that for what we have

$$F(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m)=1,$$

where F is the Lagrangean of the functional J from the formula (2.2). In this case we have $J = T$ and then the optimality of the command $u(t)$ reduces to finding the minimum time in which the mobile system arrives from the position x_0 in the position x_1. We say that we have a control "in optimal time".

We come back to the system (10.2.1) which can be rewritten in the vectorial form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), u(t)), \tag{10.2.6}$$

in which $f(x, ut)$ is the vector field of the components $f_1(x, u)$, $f_2(x, u)$, ..., $f_n(x, u)$. The functions f_i depend on the vector field of the state x and on the vector field of control u. In the form (2.1), or in the form (2.6) of the system, on the right-hand side, does not appear t, as explicit variable, such that we say that this system is an autonomous system of differential equations.

If we have a known value for the control $u(t)$, said $u_0(t)$, then the system (10.2.6) receives the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), u_0(t)),$$

which is an usual system of differential equations.

Moreover, if we impose the initial condition $x(0) = x_0$, in the usual conditions of regularity imposed to the vectorial function f, (see, for instance, Picard's theorem), we get an unique solution.

Usually, the functions f_i are assumed be continuous with respect to u and continuous differentiable with respect to x. But, in some concrete problems, the control $u = u(t)$ can have certain points of discontinuity, namely, of first species. If t_0 is such a point, then we will use the notations:

$$u\left(t_0 - 0\right) = \lim_{t \nearrow t_0} u(t), \ u\left(t_0 + 0\right) = \lim_{t \searrow t_0} u(t).$$

For comfortableness, we make the convention that the value of the control in a point of discontinuity of first species is equal to the value of the limit at left:

$$u\left(t_0\right) = u\left(t_0 - 0\right). \tag{10.2.7}$$

For $t \in [0, T]$ we say that $u(t)$ is an *accessible control* if it satisfies the conditions:

- u is continuous, except a finite number of points;
- u satisfies Eq. (2.7) in the points of discontinuity (of first species);
- u is continuous at the ends of the interval.

10.3 Linear Problems of Control

In the linear case, the evolution of a mobile system can be modeled by the following Cauchy's problem:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t), \\ x(0) = x_0, \end{cases} \tag{10.3.1}$$

where, for $t \in [0, T]$, $A(t)$ is a matrix whose components are measurable and essential bounded functions, that is $A = \left(a_{ij}\right)$, $a_{ij} = a_{ij}(t) \in L^\infty(0, T)$. Also, $B(t)$ is a matrix, $B = \left(b_{ij}\right)$, $b_{ij} = b_{ij}(t) \in C(0, T)$. The function $f = f(t)$ is given and we have $f \in L^1(0, T, R^n)$. The function $u = u(t)$ is the function of control or command and it is a fixed function and $u \in L^1(0, T, R^m)$. Finally, the function of the state $x = x(t)$ is the unknown function.

In the above conditions of regularity regarding the coefficients of the system (10.3.1), the Cauchy's problem (10.3.1) admits an unique solution which is an absolute continuous function.

In fact, since we have considered the function $u(t)$ as a fixed function, we deduce that the problem (10.3.1) is just the standard Cauchy's problem:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

and it is well known that this problem has a unique solution, if f is a continuous function with respect to the variables t and x and f is a Lipschitzian function with regard to the variable x (according to classical theorem due to Picard). On the other hand, a theorem due to Caratheodory assures that if f is a continuous function with respect to x and a measurable function with respect to t and if $\|f(t,x)\| \le g(t)$, where $g \in L^1(0,T)$, the above standard Cauchy's problem has a local solution which is an absolute continuous function perhaps defined on $[0,T]$. In addition, if f is a Lipschitzian function with respect to x then we obtain the uniqueness of the solution.

We now return back to the problem (10.3.1) and suppose that $A(t)$ is a matrix of continuous functions. As it is known, in this situation there exists a fundamental matrix of linear independent solutions, denoted by $\Phi(t)$ such that the solution of the problem (10.3.1) can be represented with the help of $\Phi(t)$ in the form:

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)\,[B(s)u(s) + f(s)]\,\mathrm{d}s,\ \ t \in [0,T]. \tag{10.3.2}$$

It is easy to verify that the function $x(t)$ is an absolute continuous function. We shall use the notation $V(t,s) = \Phi(t)\Phi^{-1}(s)$ and remember that $V(t,s)$ is the *matrix of transition* of the differential system from the problem (3.1). It is easy to see that the solution (10.3.2) can be written in the form:

$$x(t) = \Phi(t)\left[\Phi^{-1}(0)x_0 + \int_0^t \Phi^{-1}(s)\,[B(s)u(s) + f(s)]\,\mathrm{d}s\right],\ \ t \in [0,T].$$

In order to deduce this formula, or, its equivalent form (10.3.2), we can use the well known manner: First, it is attached the homogeneous form of the vectorial equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = A(t)\Phi(t),$$

and then it is attached the initial condition $\Phi(0) = I$, where I is the unit matrix.

A very important problem regarding the differential system (10.3.1) is the problem of the *controllability* which, essentially, means the following problem: given the state x_0, we want to know if we can arrive to another state x_1, which is pre-established, through one of the trajectories of the system (10.3.1), that is, if exists a control u which determines the solution of the system to arrive in x_1. If exists such a control, it is said that the pre-established state x_1 is *accessible* from the state x_0.

Finally, we specify that *the stability* of the system (10.3.1) means that there exists a control u for which the system (10.3.1) considered only in the unknown function $x(t)$, (therefore, the components of the control $u(t)$ are considered as parameters) is stable in the classical sense of the differential equations and of the systems of differential equations.

Definition 10.3.1 We say that the system (10.3.1) is controllable in the time T if for any two states x_0 and x_1, where $x_0, \ x_1 \in R^n$, there exists the control $u \in L^1(0, T, R^m)$ such that the solution of the system (10.3.1), corresponding to this u, verifies the conditions $x(0) = x_0$ and $x(T) = x_1$.

Definition 10.3.2 We say that the system (10.3.1) is null controllable in the time T if it is controllable starting from the origin, that is we have the definition 3.1 with $x_0 = 0$.

It is easy to prove that a controllable system is null controllable too. Although it seems to be surprising, it is a true conversely result.

Proposition 10.3.1 *If a system is null controllable, then it is controllable.*

Proof Let us consider an arbitrary state x_0 and let us prove that there exists a control $u \in L^1(0, T, R^m)$ which determines the system, which starts from the state x_0, to arrive, in the time T, in the state x_1 where it should arrive if it starts from the origin. If we take into account the form (10.3.2) of the solution of the system (10.3.1), we should obtain:

$$x_1 = x(T) = \Phi(T)\Phi^{-1}(0)x_0 + \int_0^T \Phi(T)\Phi^{-1}(s)\,[B(s)u(s) + f(s)]\,\mathrm{d}s,$$

such that we can say that x_1 is accessible. But this relation can be written in the form:

$$\int_0^T \Phi(T)\Phi^{-1}(s)\,[B(s)u(s) + f(s)]\,\mathrm{d}s = x_1 - \Phi(T)\Phi^{-1}(0)x_0. \quad (10.3.3)$$

But, according to hypothesis that the system is null controllable, that is, starting from the origin any state is accessible, in particular, also the state

$$x^* = x_1 - \Phi(T)\Phi^{-1}(0)x_0,$$

is accessible, that is the relation (10.3.3) is still valid and, therefore, the state x_1 is attained in the time interval T, starting from the initial state x_0 and the proof of the proposition is closed. ■

Observation 10.3.1 *The controllability of the system (10.3.1) is equivalent with the following equality of sets:*

$$\left\{ \int_0^T \Phi(T)\Phi^{-1}(s)B(s)u(s)\mathrm{d}s + \int_0^T \Phi(T)\Phi^{-1}(s)f(s)\mathrm{d}s, \ u \in L^1(0, T, R^m) \right\} = R^n.$$

Indeed, first let us observe that this equality can be written, equivalently, in the form:

$$R^n = \left\{ \int_0^T \Phi(T)\Phi^{-1}(s)B(s)u(s)\mathrm{d}s,\ u \in L^1(0, T, R^m) \right\}, \tag{10.3.4}$$

or

$$R^n = \left\{ \int_0^T \Phi^{-1}(s)B(s)u(s)\mathrm{d}s,\ u \in L^1(0, T, R^m) \right\}.$$

It is easy to establish that the inclusion $\{\} \subset R^n$ is still valid even if the system is not controllable because in the bracket $\{\}$ we have elements from R^n. Then the inclusion $R^n \subset \{\}$ is assured by the definition of the null controllable system.

Another characterization of the controllability is given in the following theorem.

Theorem 10.3.1 *The system (10.3.1) is controllable in the time interval T if and only if from the equality*

$$B^*(t)\left(\Phi^*\right)^{-1}(t)x_0 = 0,\ \forall t \in [0, T],$$

must necessarily result $x_0 = 0$, where by B^* and Φ^* we have denoted the adjuncts of the matrix B and Φ.

Proof First, let us remember a result from the theory of the Hilbert spaces: If X is a Hilbert space of finite dimension then one of its subspace X_0 (which, evidently is of finite dimension too) coincides with the whole space X if $x = 0$ is the single vector which is orthogonal on the subspace X_0. In view of the proof of our theorem we will use the characterization (10.3.4) of the controllability. Consider the subspace of R^n given by

$$X_0 = \left\{ \int_0^T \Phi^{-1}(s)B(s)u(s)\mathrm{d}s,\ u \in L^1(0, T, R^m) \right\}.$$

It is obvious that $X_0 \subset R^n$. In order to have the equality $X_0 = R^n$, according to the above result from the theory of the Hilbert spaces, we must demonstrate the implication

$$x_0 \perp X_0 \Rightarrow x_0 = 0.$$

But the orthogonality relation $x_0 \perp X_0$ in our case becomes:

$$\int_0^T \left(\Phi^{-1}(s)B(s)u(s), x_0\right) \mathrm{d}s = 0,\ \forall u \in L^1(0, T, R^m),$$

which can be written in the equivalent form:

$$\int_0^T \left(u(s)B^*(s)\left(\Phi^*\right)^{-1}(s), x_0\right) \mathrm{d}s = 0,\ \forall u \in L^1(0, T, R^m).$$

Because this relation is still valid for any u, we deduce that this relation is also still valid for

$$u(s) = B^*(s)\left(\Phi^*\right)^{-1}(s)x_0$$

too, and then we obtain:

$$\int_0^T \left\| B^*(s)\left(\Phi^*\right)^{-1}(s)x_0 \right\| \mathrm{d}s = 0 \Rightarrow B^*(s)\left(\Phi^*\right)^{-1}(s)x_0 = 0 \Rightarrow x_0 = 0,$$

such that the proof of the theorem is closed. ■

In the particular case when the matrices $A(t)$ and $B(t)$ from the linear system of control (10.3.1) are constant, i.e. $A(t) = A$ and $B(t) = B$, a new and easier characterization of the controllability is obtained due to Kalmann.

Theorem 10.3.2 *The linear system of control (10.3.1), in which $A(t) = A$ and $B(t) = B$, is controllable if and only if*

$$\operatorname{rank}\left\| B,\ AB,\ A^2B, \ldots, A^{n-1}B \right\| = n. \tag{10.3.5}$$

Proof From the classical theory of the systems of differential equations it is known that if the matrix of the coefficients of such a system is constant, then the fundamental matrix of the solutions $\Phi(t)$ is $\Phi(t) = e^{At}$. So, we deduce that $\Phi^*(t) = e^{A^*t}$ and $(\Phi^*)^{-1}(t) = e^{-A^*t}$. Then the condition of controllability from Theorem 10.3.1 becomes

$$B^*e^{-A^*t}x_0 = 0 \Rightarrow x_0 = 0.$$

If we write the series of powers for the function e^{-A^*t}, we obtain

$$e^{-A^*t} = I - \frac{t}{1!}A^* + \frac{t^2}{2!}\left(A^*\right)^2 - \ldots + (-1)^{n-1}\frac{t^{n-1}}{(n-1)!}\left(A^*\right)^{n-1} + (-1)^n\frac{t^n}{(n)!}\left(A^*\right)^n + \ldots$$

Then, the equality $B^*e^{-A^*t}x_0 = 0 \Rightarrow x_0 = 0$, $\forall t \in [0, T]$ can be written in the form:

$$\begin{cases} B^*x_0 = 0 \\ B^*A^*x_0 = 0 \\ \vdots \\ B^*\left(A^*\right)^{n-1} x_0 = 0 \\ B^*\left(A^*\right)^n x_0 = 0 \\ \vdots \end{cases} \tag{10.3.6}$$

In this system only the first n equations are essential because the others can be obtained from the first by linear combinations. For instance, the matrix $(A^*)^n$ is a combination of the powers $(A^*)^k$, in which $k \le n-1$.
Indeed, based on the known Hamilton–Cayley theorem, any matrix verifies its proper characteristic equation:

$$\left(A^*\right)^n + \lambda_1 \left(A^*\right)^{n-1} + \dots + \lambda_{k-1}A^* + I = 0$$

and from this equality we deduce:

$$\left(A^*\right)^n = -\left[\lambda_1 \left(A^*\right)^{n-1} + \lambda_2 \left(A^*\right)^{n-2} + \lambda_{k-1}A^* + I\right].$$

Therefore, except the first n equations from (10.3.6), the others equations are identically satisfied if the first n are satisfied. So, we take into consideration only the system of the first n equations from (10.3.6), that is:

$$\begin{cases} B^*x_0 = 0 \\ B^*A^*x_0 = 0 \\ \vdots \\ B^*\left(A^*\right)^{n-1} x_0 = 0. \end{cases} \tag{10.3.7}$$

But in Eq. (10.3.7) we have a linear and homogeneous system and then it admits only the null solution if and only if the rank of the matrix of the coefficients is n, that is the number of the unknown functions. With other words, we have

$$\operatorname{rank} \left\| B^*,\ B^*A^*,\ B^*\left(A^*\right)^2, \dots, B^*\left(A^*\right)^{n-1} \right\| = n,$$

which is equivalently to

$$\operatorname{rank} \left\| B,\ AB,\ A^2B, \dots, A^{n-1}B \right\| = n$$

that closes the proof of the theorem. ■

We return our attention to the system of linear control (10.3.1) and consider it now in its homogeneous form:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), \\ x(0) = x_0. \end{cases} \tag{10.3.8}$$

Suppose that $\forall x_0, x_1 \in R^n$ there exists a control $u(t)$ such that the corresponding solution of the system (10.3.8) leads x_0 in x_1, that is x_1 is accessible from x_0. We denote by x_u the solution of the system (10.3.8) corresponding to the control u and then we have:

$$x_u(0) = x_0, \;\; x_u(T) = x_1. \tag{10.3.9}$$

We will always suppose that $u \in \Omega$ where Ω is a convex, closed and bounded set from $L^\infty(0, T, R^m)$.

The matrix of transition or transference, which has been already defined, is $U(t, s) = \Phi(t)\Phi^{-1}(s)$. We write the solution of the system (10.3.8) corresponding to the control u with the help of the matrix of transition:

$$x_u(t, x_0) = U(t, 0)x_0 + \int_0^T U(t, s)B(s)u(s)\mathrm{d}s. \tag{10.3.10}$$

The set $K(T) = \{x_u(T, x_0) : \; u(t) \in \Omega, \;$ almost everywhere $t \in [0, T]\}$ is called *the set of tangibility* for the linear system of control.

The essential properties of the set of tangibility are contained in the following theorem:

Theorem 10.3.3 *The set of tangibility $K(T) \subset R^n$ is a convex and compact set.*

Proof First, we demonstrate the convexity. Let us consider $\forall x_1, x_2 \in K(T)$ and $\forall \lambda \in [0, 1]$. We must show that $\lambda x_1 + (1-\lambda)x_2 \in K(T)$. But, because $x_1, x_2 \in K(T)$, we deduce that $\exists u_1, u_2 \in \Omega$ such that

$$x_1 = U(t, 0)x_0 + \int_0^T U(t, s)B(s)u_1(s)\mathrm{d}s$$

and

$$x_2 = U(t, 0)x_0 + \int_0^T U(t, s)B(s)u_2(s)\mathrm{d}s.$$

then

$$\lambda x_1 + (1-\lambda)x_2 = \lambda U(t, 0)x_0 + (1-\lambda)U(t, 0)x_0 +$$

$$+\lambda \int_0^T U(t, s)B(s)u_1(s)\mathrm{d}s + (1-\lambda)\int_0^T U(t, s)B(s)u_2(s)\mathrm{d}s =$$

$$= U(t,0)x_0 + \int_0^T U(t,s)B(s)\,[\lambda u_1(s) + (1-\lambda)u_2(s)]\,\mathrm{d}s.$$

But Ω is a convex set and therefore we have that $\lambda u_1 + (1-\lambda)u_2 \in \Omega$. So, from the last equality we deduce that $\lambda x_1 + (1-\lambda)x_2 \in K(T)$. Let us now demonstrate the fact that the set $K(T)$ is bounded. For this, we will use the fact that the matrix of transition is a bounded matrix, $\|U(t,s)\| \leq M, \ \forall t,s \in [0,T]$, the set Ω is also bounded too, and, by hypothesis, for the matrix B we have $B \in L^\infty(0,T)$. Then

$$\|x_u(T,x_0)\| \leq \|U(T,0)\|\,\|x_0\| + \int_0^T \|U(T,s)\|\|B(s)\|\|u(s)\|\mathrm{d}s.$$

Finally, we will show that the set $K(T)$ is closed, by using the characterization with the help of sequences. Let $\{y_n\}_{n\in N}$ be an arbitrary sequence of elements from $K(T)$ such that $y_n \to y$. If we show that $y \in K(T)$, by using the characterization by sequences of a closed set, we will deduce that the set $K(T)$ is closed. First, because $y_n \in K(T), \ \forall n \in N$, we deduce, from the definition of the set $K(T)$, that there exists $u_n \in \Omega$ such that

$$y_n = U(t,0)x_0 + \int_0^T U(t,s)B(s)u_n(s)\mathrm{d}s. \tag{10.3.11}$$

In functional analysis it is proved the following result:
Any set from the dual of a Banach space is a weak compact set, that is from any sequence of elements from such a set we can extract a subsequence which is weak convergent (see Sect. 10.1).

In our case, $u_n \in \Omega \subset L^\infty$ and L^∞ is the dual of the Banach space L^1. On the other hand, $\Omega \subset L^\infty$ is bounded and $u_n \in \Omega$ and then, based on the above result from functional analysis, we deduce that we can extract the subsequence u_{n_k} which is weak convergent, say to $u \in \Omega$. Therefore $u_{n_k}(s) \rightharpoonup u(s)$, that is

$$\int_0^T u_{n_k}(s)f(s)\mathrm{d}s \rightharpoonup \int_0^T u(s)f(s)\mathrm{d}s.$$

Corresponding to the subsequence u_{n_k} we have the subsequence y_{n_k}. Now we write the relation (10.3.11) corresponding to the subsequences y_{n_k} and, respectivelly, u_{n_k} and we obtain that $y_{n_k} \rightharpoonup y$. But we have assumed that the sequence y_n is convergent to y and, therefore, any of its subsequence is convergent to the same limit, that is $y_n \to y$ and the proof of the theorem is closed. ∎

In all our previous considerations we have defined and characterized a control. In all what follows, we shall demonstrate the existence of an optimal control.

We call *optimal control* that control for what the corresponding solution of the system (10.3.8) arrives in the state x_1 in a minimum time.

Theorem 10.3.4 *Consider the state $x_1 \in R^n$ having the property that there exists $T > 0$ and an admissible control u such that $x_u(T, x_0) = x_1$. Then, there exists a minimum time T^* and a corresponding control u^* (therefore, an optimal control) such that $x_{u^*}(T^*, x_0) = x_1$.*

Proof For x_1 having the property from the hypothesis we have that $x_1 \in K(T)$. We denote by T^* the following infimum:

$$T^* = \inf \{T : x_1 \in K(T)\}. \tag{10.3.12}$$

We intend to demonstrate that $x_1 \in K(T^*)$. From the definition of T^* we deduce that there exists a sequence $\{T_n\}_{n\in N}$ such that $T_n \to T^*$ and then evidently $x_1 \in K(T_n)$. We denote by u_n^* the control which corresponds to T_n. We have, therefore

$$u_n^*(s) = \begin{cases} u_n(s), & s \in [0, T_n], \\ 0, & s \in (T_n, T]. \end{cases}$$

As a consequence, we can write:

$$x_1 = U(T_n, 0)x_0 + \int_0^{T_n} U(T_n, s)B(s)u_n^*(s)\mathrm{d}s. \tag{10.3.13}$$

By using the above form of $u_n^*(s)$, the relation (10.3.13) receives the form:

$$x_1 = U(T_n, 0)x_0 + \int_0^{T_n} U(T_n, s)B(s)u_n(s)\mathrm{d}s.$$

The sequence $u_n^*(s)$ is bounded in L^∞ and, therefore, it contains a subsequence which is weak convergent to an element u from L^∞. Then, without restriction of the generality, we suppose that

$$u_n^* \rightharpoonup u^* \text{ in } L^\infty(0, T, R^n).$$

Therefore, we deduce that u^* is an admissibil control and $u^* = 0$ for $t \in [T^*, T]$. Passing now to the limit in Eq. (10.3.13), we obtain:

$$x_1 = U(T^*, 0)x_0 + \int_0^{T^*} U(T^*, s)B(s)u^*(s)\mathrm{d}s,$$

and this shows that $x_1 \in K(T^*)$ such that the proof of the theorem is closed. ■

In the following theorem we will give a characterization of the optimal control and of the corresponding optimal time. This result is known having the name *the maximum principle* of Pontreaghin.

Theorem 10.3.5 *Consider a problem of the minimum type. Let* $(x^*,\ u^*,\ T^*)$ *be an optimal third form, that is the optimal state, the optimal control and the optimal time. Then, there exists* η^* *and* η_1 *which verify the system:*

$$\begin{cases} \frac{d}{dt}(x^*) = A(t)x^* + B(t)u^* \\ x^*(0) = x_0 \\ x^*(T^*) = x_1 \\ \frac{d}{dt}(\eta^*) = -A^*(t)\eta^* \\ \eta^*(T^*) = \eta_1 \end{cases} \tag{10.3.14}$$

and, in addition, the relation:

$$B^*(t)\eta^*(t) \in \partial I_\Omega\left(u^*(t)\right),\ \text{almost everywhere } t \in [0, T^*]. \tag{10.3.15}$$

Conversely, if there exists η_1 *and* η *which verify the system (10.3.14) and the relation (10.3.15), then the third form* $(x^*,\ u^*,\ T^*)$ *is an optimal third form.*

Proof First, we must specify that I_Ω represents *the indicator function* of the set Ω which is assumed be a convex and closed set, that is

$$I_\Omega = \begin{cases} 0, & t \in \Omega, \\ +\infty, & t \notin \Omega. \end{cases}$$

It is easy to show that the indicator function is a convex and s. c. i. function. Also, I_Ω is a subdifferentiable function, therefore it exists ∂I_Ω. According to the definition of the subdifferential, the relation (10.3.15) can be written, equivalently, in the form:

$$\left(B^*\eta^*, u^* - u\right) \geq 0,\ \forall u \in \Omega,\ \text{almost everywhere } t \in [0, T].$$

The necessity. Suppose that the third form $(x^*,\ u^*,\ T^*)$ is optimal. By accommodating the definition of the set of tangibility, we have

$$K(T^*) = \left\{x_u\left(T^*, x_0\right) :\ u \in \Omega \text{ almost everywhere } t \in [0, T^*]\right\}.$$

First, we demonstrate that

$$x_1 \in \text{Fr } K(T^*), \tag{10.3.16}$$

where $\text{Fr } K(T^*)$ is the border of the set $K(T^*)$.

Suppose, by absurdum, that $x_1 \notin \text{Fr}K(T^*)$. Then $\forall\varepsilon > 0$ we have that $x_1 + w \in K(T^*)$ for all $w \in R^n$ having the property $\|w\| < \varepsilon$. Therefore $x_1 + w$ has the form

$$x_1 + w = U(T^*, 0)x_0 + \int_0^{T^*} U(T^*, s)B(s)u(s)\mathrm{d}s,$$

where u is an admissible control. We will write x_1 in the form

$$x_1 = U(T^* - \delta, 0)x_0 + \int_0^{T^*-\delta} U(T^* - \delta, s)B(s)u^*(s)\mathrm{d}s +$$

$$+ \int_0^{T^*-\delta} \left[U(T^*, s) - U(T^* - \delta, s)\right] B(s)u^*(s) +$$

$$+ \left[U(T^*, 0) - U(T^* - \delta, 0)\right] x_0 + \int_{T^*-\delta}^{T^*} U(T^*, s)B(s)u^*(s)\mathrm{d}s.$$

Therefore, x_1 is of the form $x_1 = y_1 - w$, where $y_1 \in K(T^* - \delta)$ and w has the property $\|w\| < \varepsilon$ from where we deduce that $x_1 + w = y_1 \in K(T^* - \delta)$ with $\|w\| < \varepsilon$. But this means that $x_1 \in K(T^* - \delta)$ which is in the contradiction with the fact that T^* is optimal, that is minimum. This contradiction demonstrates that $x_1 \in FrK(T^*)$. Then, there exists a hyperplane of support for the set $K(T^*)$ in the point x_1. We denote by η_1 the normal to this hyperplane. This normal can be characterized by:

$$(\eta_1, x_1 - x) \geq 0,\ \forall x \in K(T^*) \Rightarrow \left(\eta_1, x_1 - x_u\left(T^*, x_0\right)\right) \geq 0,\ \forall u \in \Omega \Rightarrow$$

$$\Rightarrow \left(\eta_1, \int_0^{T^*} U(T^*, s)B(s)[u^*(s)u(s)]\mathrm{d}s\right) \geq 0,\ \forall u \in \Omega \Rightarrow$$

$$\Rightarrow \int_0^{T^*} \left(u^*(s) - u(s),\, B^*(s)U^*(T^*, s)\eta_1\right) \mathrm{d}s \geq 0,\ \forall u \in \Omega \Rightarrow$$

$$\Rightarrow \left(u^*(s) - u(s), B^*(s)U^*(T^*, s)\eta_1\right) \geq 0,\ \forall u \in \Omega,\ \textit{almost everywhere } s \in [0, T^*].$$

We use the notation $\eta^* = U^*(T^*, s)\eta_1$ and then, by taking into account the definition of the subdifferential, the last relation can be written in the form:

$$B^*(t)\eta^*(t) \in \partial I_\Omega\left(u^*(t)\right).$$

Since, by definition, we have $U(T,s) = \Phi(T)\Phi^{-1}(s)$, we deduce that

$$U^*(T^*, s) = \left(\Phi^*\right)^{-1}(s)\Phi^*(T^*). \tag{10.3.17}$$

Now, we intend to find the fundamental matrix for the equation

$$\frac{d}{\mathrm{d}t}\left(\eta^*\right) = -A^*\eta^*.$$

In fact, we will show that Φ^* is the fundamental matrix for this equation, knowing that Φ is the fundamental matrix in the case of the equation having the matrix A.

By using the definition of η^* let us show that $\eta^*(T^*) = \eta_1$. Taking into account the form of U, we have:

$$\eta^*\left(T^*\right) = U\left(T^*, T^*\right)\eta_1 = \Phi\left(T^*\right)\Phi^{-1}\left(T^*\right)\eta_1 = \eta_1.$$

With the help of the relation (10.3.17) we will deduce that

$$\frac{d}{\mathrm{d}t}\left(\eta^*\right) = -A^*\eta^*.$$

Indeed, we have

$$\frac{d}{\mathrm{d}s}\left(\eta^*(s)\right) = \frac{d}{\mathrm{d}s}\left(U^*\left(T^*, s\right)\eta_1\right) = \frac{d}{\mathrm{d}s}\left(\Phi^*(s)\right) = \left(\Phi^*(s)\right)^{-1}\Phi^*(s)\left(\Phi^*(s)\right)^{-1}.$$

But $\Phi^*(s)\left(\Phi^*(s)\right)^{-1} = I$ such that we deduce that

$$\frac{d}{\mathrm{d}s}\left(\Phi^*(s)\right)\left(\Phi^*(s)\right)^{-1} + \Phi^*(s)\frac{d}{\mathrm{d}s}\left(\Phi^*(s)\right)^{-1} = 0.$$

Now, we use the fact that $(\Phi^*)^{-1} = A * \Phi^*(s)$ and therefore we can write

$$\frac{d}{\mathrm{d}s}\left(\eta^*(s)\right) = -\left(\Phi^*(s)\right)^{-1}A^*(s)\Phi^*(T) = -A^*(s)\eta^*(s),$$

which is our desired result. The proof of the necessity is closed.

Sufficiency. We can follow the reverse line from the proof of the necessity. However, we relate in detail the fact that η^* verifies the equation:

$$\frac{d}{\mathrm{d}t}\left(\eta^*\right) = -A^*\eta^*.$$

We have

$$\frac{d}{\mathrm{d}s}\left(\eta^*(s)\right) = \frac{d}{\mathrm{d}s}\left(U^*\left(T^*, s\right)\eta_1\right) = \frac{d}{\mathrm{d}s}\left[\left(\Phi^*(s)\right)^{-1}\Phi^*(T^*)\eta_1\right] =$$

$$= \frac{d}{ds}\left[\left(\Phi^*(s)\right)^{-1}\right]\Phi^*(T^*)\eta_1 = \frac{d}{ds}\left[\left(\Phi^*(s)\right)^{-1}\right]\Phi^*(s)\left[\left(\Phi^*(s)\right)^{-1}\Phi^*(T^*)\right]\eta_1.$$

Taking into account that $(\Phi^*(s))^{-1}\,\Phi^*(T^*) = U^*(T^*, s)$, we obtain

$$\frac{d}{ds}\left(\eta^*(s)\right) = \frac{d}{ds}\left[\left(\Phi^*(s)\right)^{-1}\right]\Phi^*(s)U^*(T^*, s)\eta_1 = \frac{d}{ds}\left[\left(\Phi^*(s)\right)^{-1}\right]\Phi^*(s)\eta^*.$$

But $(\Phi^*(s))' = A^*\Phi^*$ and

$$\frac{d}{ds}\left[\left(\Phi^*(s)\right)^{-1}\right]\Phi^*(s) = -\frac{d}{ds}\left[\Phi^*(s)\right]\left(\Phi^*(s)\right)^{-1}.$$

In conclusion, we have:

$$\frac{d}{ds}\left(\eta^*(s)\right) = -\frac{d}{ds}\left(\Phi^*(s)\right)\left(\Phi^*(s)\right)^{-1}\eta^* = -A^*\Phi^*\left(\Phi^*(s)\right)^{-1}\eta^* = -A^*\eta^*,$$

and the proof of the theorem is closed. ■

Remarks. **1**. As a first conclusion which follows from this theorem is that, now we can compute, in the conditions of the theorem, both the optimal control and also optimal time too.

2. The relation (3.15) shows that if $u^*(t) \in Int\Omega$ then $B^*(t)\eta^*(t) = 0$ and if $B^*(t)\eta^*(t) \neq 0, \forall t \in [0, T^*)$ then $u \in Fr\Omega, \forall t \in [0, T^*)$.

Application. We will conclude this paragraph by giving a concrete application. Consider the system of linear control:

$$\begin{cases} x' = x + u \\ x(0) = x_0 \end{cases}$$

and as a domain of control we consider the set $\Omega = \{u : |u| < 1\}$.

We intend to determine the optimal control which leads 0 in 1. By using Eq. (10.3.14) we obtain the system

$$\begin{cases} \eta' = -\eta, \\ \eta(T^*) = \eta_1. \end{cases}$$

In our case, the relation (10.3.15) becomes: $B^*\eta^* \in \partial I_\Omega(u^*)$ which can be written, equivalently, in the form:

$$u^* \in (\partial I_\Omega)^{-1}\left(B^*\eta^*\right).$$

But $(\partial I_\Omega)^{-1} = \partial I_\Omega^*$. Then we have that $I_\Omega^*(u) = \|u\|$ and then $\partial I_\Omega^*(u) = \partial\|u\|$. So, we obtain:

$$u^* \in (\partial I_\Omega)^*\left(B^*\eta^*\right) = \partial\|B^*\eta^*\| = \text{sign}\left(B^*\eta^*\right).$$

In the present case, we have

$$u^* = \text{sign } \eta^* = \frac{\eta^*}{|\eta^*|}.$$

In conclusion, we must solve the system

$$\begin{cases} (x^*)' = x^* + u^* \\ x^*(0) = 0 \\ x^*(T^*) = 1 \\ (\eta^*)' = -\eta^* \\ \eta^*(T^*) = \eta_1. \end{cases}$$

By using the first two relations, we find

$$x^* = \int_0^t e^{t-s} u^*(s) \mathrm{d}s.$$

Using the last condition in the last two equations we obtain the solution:

$$\eta^*(t) = \eta_1 e^{T^*-1}.$$

First, we suppose that $\eta_1 > 0$ such that we have $\eta^* > 0$, therefore $u^* = \eta^*/|\eta^*| = 1$ such that the above solution becomes

$$x^*(t) = \int_0^t e^{t-s} \mathrm{d}s = e^t - 1.$$

We impose to this solution to satisfy the condition $x^*(T^*) = 1$ such that we obtain $e^{T^*} - 1 = 1$ and then $T^* = \ln 2$. If we suppose that $\eta_1 < 0$ then and $\eta^* < 0$ and then we can follow the above procedure by using $u^* = -1$ such that we will obtain $x^*(t) = 1 - e^t$. If we impose the condition $x^*(T^*) = 1$ we are led to $1 - e^{T^*} = 1$, from where we deduce that $e^{T^*} = 0$ which is absurdum. Therefore always we have $\eta_1 > 0$.

10.4 Problems of Quadratic Control

Let us consider the system of control

$$\begin{cases} x'(t) = A(t)x(t) + B(t)u(t) + f(t) \\ x(0) = x_0. \end{cases} \tag{10.4.1}$$

We now attach the problem of quadratic minimum:

$$\min \left\{ \int_0^T [(Q(t)x(t), x(t)) + (N(t)u(t), u(t))] \, dt, \ u(t) \in \Omega \right\}, \quad (10.4.2)$$

where Ω is the set of admissibility which is assumed be a closed, convex and bounded set. The matrices A and B are quadratic and have the elements from L^∞, and the function on the right-hand side $f(t)$ is assumed to be an integrable function. Also, the quadratic matrices Q and N have the elements from L^∞ and are assumed to be symmetric and positive definite, therefore

$$(Qx, x) \geq w\|x\|^2, \ \forall x \in R^n.$$

A similar relation satisfies also the matrix N too.

We will demonstrate that, based on these hypotheses the problem of minimum (10.4.2) has a solution and this is even unique.

Theorem 10.4.1 *There exists a unique control u and a unique trajectory of the system (4.1) which verify the problem (10.4.2).*

Proof For the proof we will use an already remembered result in the introductory paragraph:

If the function $\varphi : X \to R \cup \{\infty\}$ is convex and s. c. i. and, in addition, satisfies the condition

$$\lim_{\|x\| \to \infty} \varphi(x) = \infty,$$

then φ attains its effective minimum on the Banach space X.

If, in addition, the function φ is strict convex, then its minimum is unique. The function φ is strict convex if:

$$\varphi(\lambda x + (1 - \lambda)y < \lambda\varphi(x) + (1 - \lambda)\varphi(y), \ \forall x, y \in X, \ x \neq y, \ \forall \lambda \in (0, 1).$$

For the proof of our theorem we will construct a function which satisfies the above mentioned conditions and then this will have an unique minimum which will be the minimum of the problem (10.4.2). We define the function φ by:

$$\varphi(u) = \begin{cases} \int_0^T [(Qx_u, x_u) + (Nu, u)] \, dt, \ u(t) \in \Omega, \ \text{almost everywhere } t \in [0, T] \\ +\infty, \qquad \text{elsewhere} \end{cases} \quad (10.4.3)$$

Here x_u represents the solution of the system (10.4.1) corresponding to the control u, namely

$$x_u(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^T \Phi(t)\Phi^{-1}(s)B(s)u(s)ds.$$

We introduce, as usual, the matrix of transition $U(t, s) = \Phi(t)\Phi^{-1}(s)$ and then the solution x_u can be written in the form:

$$x_u(t) = U(t, 0)x_0 + \int_0^T U(t, s)B(s)u(s)\mathrm{d}s.$$

We take $\Omega \subset L^2$ and then we will consider that the function φ is defined on the space $L^2(0, T, R^n)$, therefore

$$\varphi : L^2\left(0, T, R^n\right) \to R \cup \{\infty\}.$$

The fact that φ is a convex function can be demonstrated without any difficulty. Let us show that φ is a semi-continuous inferior function. For this we will use the fact that a bounded set of values of the function φ is closed. Consider the bounded set $A = \left\{u \in L^2 : \varphi(u) \le M\right\} \subset L^2$ and let us show that A is a closed set. Let $\{u_n\}_{n\in N}$ be a sequence of elements from A assumed to be convergent, in the sens of the norm of the space L^2, to an element u. Because $u_n \in A$ we deduce that $\varphi(u_n) \le M$ and if we take into account the definition of the function φ, we obtain that $u_n \in \Omega$ almost everywhere $t \in [0, T]$, therefore:

$$\int_0^T \left[\left(Qx_{u_n}, x_{u_n}\right) + (Nu_n, u_n)\right]\mathrm{d}t \le M. \tag{10.4.4}$$

Since the set Ω is closed and $u_n \in \Omega$ we deduce that there exists a subsequence u_{n_k} of elements from Ω which is convergent almost everywhere to $u_0 \in \Omega$. Then $x_{u_n} \to x_{u_0}$. Passing now to the limit in Eq. (10.4.4) and we obtain:

$$\int_0^T [(Qx_u, x_u) + (Nu, u)]\,\mathrm{d}t \le M,$$

that is $\varphi(u) \le M$. To close the proof, we must demonstrate that:

$$\lim_{\|u\|\to\infty} \varphi(u) = \infty.$$

This is obtained based on the fact that, by hypothesis, the matrices Q and N are positive defined, and therefore we can write:

$$\varphi(u) = \int_0^T [(Qx_u, x_u) + (Nu, u)]\,\mathrm{d}t \ge \int_0^T w\,\|u\|^2\,\mathrm{d}t = w\,\|u\|^2_{L^2(0,T,R^n)}\,.$$

From the last inequality it is clear that $\varphi(u) \to \infty$ if $\|u\| \to \infty$. Therefore, the function $\varphi(u)$ which has been constructed in Eq. (10.4.3) satisfies all the hypotheses of the theorem mentioned at the beginning of the proof and then the function φ attains its effective minimum and this minimum is the minimum of the problem (10.4.2), taking into account the mode used for the definition of the function φ. This minimum is effective if we will show that the function φ is strict convex. This is an immediate consequence of the fact that the matrices Q and N, from the definition of the function

φ, are positive defined and it is known the fact that any quadratic positive defined form is strict convex.

The final conclusion is that the problem (10.4.2) has a solution and this is unique and the proof of the theorem is closed. ■

We now adapt the maximum principle of Pontreaghin to the case of the quadratic control. In the following theorem, also due to Pontreaghin, we will find the necessary and sufficient conditions such that the pair (x^*, u^*) will be optimal, where x^* is the optimal state and u^* is the optimal control.

Theorem 10.4.2 *For the problem of minimum (10.4.2) the pair (x^*, u^*) is optimal if and only if there exists a function $p(t)$ which, together with x^* and u^*, verifies the system:*

$$\begin{cases} (x^*)' = Ax^* + Bu^* + f \\ p' = -A^* p + 2Qx^* \\ x^*(0) = x_0 \\ p(T) = 0, \end{cases} \tag{10.4.5}$$

as well as the relation

$$\big(B^*(t)p(t) - 2N(t)u^*(t),\ u^* - u\big) \geq 0,\ \forall u \in \Omega. \tag{10.4.6}$$

Proof **The sufficiency**. Suppose that there exists the function $p(t)$ which together with x^* and u^* verifies the system (10.4.5) and as well as the relation (10.4.6) and let us demonstrate that the pair (x^*, u^*) is optimal, that is:

$$\int_0^T \big[\big(Qx^*, x^*\big) + \big(Nu^*, u^*\big)\big]\, \mathrm{d}t \leq \int_0^T [(Qx, x) + (Nu, u)]\, \mathrm{d}t, \forall u \in L^2(0.T, R^n), \tag{10.4.7}$$

where it is implied the fact that $u \in \Omega$ and x is an arbitrary solution of the system (10.4.1).

In the proof of the sufficiency we will use the following two inequalities:

$$\frac{1}{2}\big(Qx^*, x^*\big) \leq \frac{1}{2}(Qx, x) + \big(Qx^*, x^* - x\big), \tag{10.4.8}$$

$$\frac{1}{2}\big(Nx^*, x^*\big) \leq \frac{1}{2}(Nx, x) + \big(Nx^*, x^* - x\big). \tag{10.4.9}$$

To demonstrate these two inequalities we will use the Schwarz's inequality:

$$(Qx, x) \leq \frac{1}{2}[(Qx, x) + (Qy, y)],$$

and this, in turn, follows from the inequality:

$$(Qx, y) \leq \sqrt{(Qx, x)\,(Qy, y)}.$$

Regarding the inequality (10.4.8) we have:

$$\left(Qx^*, x^*\right) = \left(Qx^*, x^* - x\right) + \left(Qx^*, x\right) \leq$$

$$\leq \left(Qx^*, x^* - x\right) + \frac{1}{2}\left[\left(Qx^*, x^*\right) + (Qx, x)\right],$$

from where, by simple calculations, we obtain:

$$\frac{1}{2}\left(Qx^*, x^*\right) \leq \left(Qx^*, x^* - x\right) + \frac{1}{2}\left(Qx, x\right).$$

Analogous, can be demonstrated the inequality (10.4.9) too. We now write the second equation from the system (10.4.5) in the form:

$$2Q(t)x^* = p' + A^*p$$

and multiply both sides of this equality by $x^* - x$. Then, we integrate the resulting equality on the interval $[0, T]$. So, we obtain:

$$2\int_0^T \left(Qx^*, x^* - x\right) \mathrm{d}t = \int_0^T \left[\left(p', x^* - x\right) + \left(A^*p, x^* - x\right)\right] \mathrm{d}t =$$

$$= \int_0^T \left(p, x^* - x\right)' \mathrm{d}t - \int_0^T \left[\left(p, \left(x^* - x\right)'\right) \mathrm{d}t + \left(p, A\left(x^* - x\right)\right)\right] \mathrm{d}t =$$

$$= -\int_0^T \left[\left(p, x^{*\prime} - x'\right) - \left(p, Ax^* - Ax\right)\right] \mathrm{d}t = -\int_0^T \left(p, x^{*\prime} - x - Ax^* + Ax\right) \mathrm{d}t =$$

$$= -\int_0^T \left(p, Ax^* + Bu^* + f - Ax^* + Ax\right) \mathrm{d}t = -\int_0^T \left(p, B\left(u^* - u\right)\right) \mathrm{d}t =$$

$$= -\int_0^T \left(B^*p, u^* - u\right) \mathrm{d}t \leq -2\int_0^T \left(Nu^*, u^* - u\right) \mathrm{d}t.$$

Therefore, we have

$$2\int_0^T \left(Qx^*, x^* - x\right) \mathrm{d}t + 2\int_0^T \left(Nu^*, u^* - u\right) \mathrm{d}t \leq 0,$$

that is:

$$\int_0^T \left[\left(Qx^*, x^* - x\right) + \left(Nu^*, u^* - u\right)\right] \mathrm{d}t \leq 0. \tag{10.4.10}$$

We now add side by side the inequalities (10.4.8) and (10.4.9), then the resulting inequality is integrated on the interval $[0, T]$. So, we obtain:

$$\frac{1}{2}\int_0^T \left(Qx^*, x^*\right) \mathrm{d}t + \frac{1}{2}\int_0^T \left(Nu^*, u^*\right) \mathrm{d}t \leq$$

$$\leq \frac{1}{2}\int_0^T [(Qx, x)+(Nu, u)]\, \mathrm{d}t+\frac{1}{2}\int_0^T \left[\left(Qx^*, x^*-x\right)+\left(Nu^*, u^*-u\right)\right] \mathrm{d}t. \tag{10.4.11}$$

Finally, from Eqs. (10.4.11) and (10.4.10) we obtain

$$\int_0^T \left[\left(Qx^*, x^*\right) + \left(Nu^*, u^*\right)\right] \mathrm{d}t \leq \int_0^T [(Qx, x) + (Nu, u)]\, \mathrm{d}t,$$

which is even the inequality (10.4.7) and the sufficiency is demonstrated.

The Necessity. Let us now suppose that the pair (x^*, u^*) is optimal and let us demonstrate that there exists a function $p(t)$ which, together with x^* and u^* verify the system (10.4.5) as well as the relation (10.4.6). In order to emphasize that the pair (x^*, u^*) is optimal, we will use the inequality:

$$\int_0^T \left[\left(Qx^*, x^*\right)+\left(Nu^*, u^*\right)\right] \mathrm{d}t \leq \int_0^T [(Qx, x)+(Nu, u)]\, \mathrm{d}t, \forall u \in L^2, u \in \Omega. \tag{10.4.12}$$

We take the control u and the state x of the form: $u = u^* + \varepsilon w$ and $x = x^* + \varepsilon z$ such that $u^* + \varepsilon w \in \Omega$ and x^* verifies the system

$$\begin{cases} (x^*)' = Ax^* + Bu^* + f \\ x^*(0) = x_0. \end{cases}$$

Then z verifies the system

$$\begin{cases} z' = Az + Bw \\ z(0) = 0. \end{cases}$$

We substitute u and x, which have the expressions above proposed, in (10.4.12) such that we obtain:

$$\int_0^T \left[\left(Qx^*, x^*\right)+\left(Nu^*, u^*\right)\right] \mathrm{d}t \leq \int_0^T \left[\left(Qx^*+\varepsilon z, x^*+\varepsilon z\right)+\left(Nu^*+\varepsilon w, u^*+\varepsilon w\right)\right] \mathrm{d}t.$$

This inequality can be written in the form:

$$\int_0^T \left[\left(Qx^*, x^*\right) + \left(Nu^*, u^*\right)\right] \mathrm{d}t \leq$$

$$\leq \int_0^T \left\{\left(Qx^*, x^*\right)+\left(Nu^*, u^*\right)+\varepsilon\left[\left(Qx^*, z\right)+\left(Nu^*, w\right)\right]+\varepsilon^2\left[(Qz, z)+(Nw, w)\right]\right\} \mathrm{d}t,$$

and this inequality is true $\forall \varepsilon > 0$. From here we deduce:

$$\int_0^T \left[\left(Qx^*, z\right) + \left(Nu^*, w\right)\right] \mathrm{d}t \geq 0, \tag{10.4.13}$$

and this inequality is true $\forall \varepsilon > 0$ such that $u^* + \varepsilon w \in \Omega$.

We will define the function $p(t)$ desired in the enunciation even the solution of the system

$$\begin{cases} p' = -A^* p + 2Qx^* \\ p(T) = 0. \end{cases}$$

We write the equation in the form $2Qx^* = p' + A^* p$, multiply both sides with z, then we integrate the resulting equality on the interval $[0, T]$:

$$\int_0^T \left(2Qx^*, z\right) \mathrm{d}t = \int_0^T \left(p' + A^* p, z\right) \mathrm{d}t = \int_0^T \left[\left(p', z\right) + \left(A^* p, z\right)\right] \mathrm{d}t =$$

$$= \int_0^T (p, z)' \, \mathrm{d}t - \int_0^T \left[\left(p, z'\right) - (p, Az)\right] \mathrm{d}t = -\int_0^T \left(p, z' - Az\right) \mathrm{d}t = -\int_0^T (p, Bw) \, \mathrm{d}t$$

From (10.4.13) we deduce:

$$\int_0^T \left[\left(2Qx^*, z\right) + \left(2Nu^*, w\right)\right] \mathrm{d}t = -\int_0^T \left[(p, Bw) - \left(2Nu^*, w\right)\right] \mathrm{d}t$$

and then we can write:

$$-\int_0^T \left[\left(B^* p, w\right) - \left(2Nu^*, w\right)\right] \mathrm{d}t \geq 0.$$

This inequality is equivalently with:

$$\int_0^T \left(2Nu^* - B^* p, w\right) \mathrm{d}t \geq 0$$

and this inequality is true for $\forall w \in L^2$ such that $u^* + \varepsilon w \in \Omega$. If we will use the notation $v = u^* + \varepsilon w \in \Omega$, then we can write

$$w = \frac{1}{\varepsilon}\left(v - u^*\right).$$

Because w has been assumed to be arbitrary, we deduce that v is arbitrary, $v \in \Omega$. Therefore, we have the inequality

$$\int_0^T \left(B^* p - 2Nu^*, u^* - v\right) \mathrm{d}t \geq 0,\ \forall v \in \Omega.$$

From here we obtain the inequality

$$\left(B^* p - 2Nu^*, u^* - v\right),\ \forall v \in \Omega,$$

and this concludes the proof. ■

At the end of this paragraph, we give a simple application of the Pontreaghin's principle in the case of a problem of quadratic control. Consider the system:

$$\begin{cases} x' = x + u \\ x(0) = 1, \end{cases} \tag{10.4.14}$$

and, as a set accessibility we will use the set Ω. We take $\Omega = [0, \infty)$. Consider the problem of minimum

$$\min \left\{ \int_0^1 \left(x^2 + u^2\right) \mathrm{d}t\ :\ x \geq 0,\ u \in \Omega \right\}. \tag{10.4.15}$$

According to the maximum principle, in order to determine the optimal pair (x^*, u^*) we must solve the system:

$$\begin{cases} x^{*\prime} = x^* + u^* \\ p' = -p + 2x^* \\ x^*(0) = 1 \\ p(1) = 0, \end{cases} \tag{10.4.16}$$

as well as the inequality

$$\left(p - 2u^*, u^* - u\right) \geq 0,\ \forall u \geq 0. \tag{10.4.17}$$

We write the inequality (10.4.17) in the form

$$p \in 2u^*(t) + \partial I_\Omega\left(u^*(t)\right) \Rightarrow p(t) \in (2I + \partial I_\Omega)\left(u^*(t)\right) \Rightarrow$$

$$\Rightarrow u^*(t) \in \left(I + \frac{1}{2}\partial I_\Omega\right)^{-1}\left(\frac{p(t)}{2}\right).$$

In all what follows we will show that the expression

$$\left(I + \frac{1}{2}\partial I_\Omega\right)^{-1} p \tag{10.4.18}$$

represents the projection of p on the set Ω.
Generally speaking, $g(t)$ is the projection of an operator f on a set Ω if:

$$|f(t) - g(t)| \leq |f(t) - \varphi(t)|, \ \forall \varphi = \varphi(t) \in \Omega.$$

In the present case, we must demonstrate that

$$\|p - y\| \leq \|p - u\|, \ \forall u \in \Omega,$$

where we have denoted by y the projection of p on Ω. Then we have:

$$p - y \in \partial I_\Omega(y) \Leftrightarrow (p - y, y - u) \geq 0, \ \forall u \in \Omega.$$

It is easy to see that this inequality can be written in the form:

$$(p - y)(y - p) \geq (p - y)(y - p) + (p - y)(p - u), \ \forall u \in \Omega.$$

From here we deduce:

$$|p - y|^2 \leq (p - y)(p - u), \ \forall u \in \Omega$$

and, after simplification

$$|p - y| \leq |p - u|, \ \forall u \in \Omega$$

that is, the desired result.

We turn now back to our application. Because Eq. (10.4.18) represents the projection of p on the set Ω, we have:

$$u^*(t) = \begin{cases} 0, & p(t) \leq 0 \\ p(t)/2, & p(t) > 0. \end{cases}$$

The case 1. If $p(t) \leq 0$ we have $u^*(t) = 0$ and then $x^*(t)$ verifies the system

$$\begin{cases} (x^*)'(t) = x^*(t) \\ x^*(0) = 1, \end{cases}$$

which evidently has the solution $x^*(t) = e^t$.

Then the function $p(t)$ verifies the system:

$$\begin{cases} p'(t) = -p(t) + 2x^*(t) \\ p(1) = 0, \end{cases}$$

which has the solution:

$$p(t) = e^{-t}\left(C + \int_0^t 2e^s \mathrm{d}s\right) = e^{-t}\left(C + e^{2t} - 1\right).$$

By using the condition $p(1) = 0$ we deduce that $C = -e^2 + 1$ and then the solution is:

$$p(t) = e^{-t}\left(e^{2t} - e^2\right).$$

The case 2. If $p(t) > 0$ then $u^*(t) = p(t)/2$. So, the optimal state $x^*(t)$ and the function $p(t)$ satisfy the following system:

$$\begin{cases} (x^*)'(t) = x^*(t) + p(t)/2 \\ p'(t) = -p(t) + 2x^*(t) \\ x^*(0) = 1 \\ p(1) = 0. \end{cases}$$

We write the characteristic equation of the system:

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ 2 & -1-\lambda \end{vmatrix} = 0,$$

that is, the equation $\lambda^2 - 2 = 0$ which has the solutions $p = \pm\sqrt{2}$. Then the solution of the system is:

$$\left| \begin{array}{l} x^*(t) = A_1 e^{\sqrt{2}t} + A_2 e^{-\sqrt{2}t} \\ p(t) = B_1 e^{\sqrt{2}t} + B_2 e^{-\sqrt{2}t}. \end{array} \right.$$

If we impose the initial conditions, $x^*(0) = 1$ and $p(1) = 0$, we obtain:

$$A_1 = e^{-2\sqrt{2}}\left(\sqrt{2}+1\right)^2, \; A_2 = 1 - e^{-2\sqrt{2}}\left(\sqrt{2}+1\right)^2.$$

10.5 The Synthesis of the Optimal Control

The problem of the synthesis of the optimal control consists of the determination of a operator Λ such that the optimal control can be expressed as a function of the optimal state $x(t)$ by a rule of the form:

$$u(t) = \Lambda(t, x(t)).$$

In this formulation, the optimal control u is expressed as a function of the present value of the optimal state. There exists also more general formulations in which u can be expressed as a function of the past values of the optimal state. If we take into account the relation which is satisfied by the optimal control, namely:

$$B^* p \in 2Nu^* + \partial I_\Omega \left(u^*\right),$$

then we deduce that in order to express the control as a function of the optimal state, it is sufficient to express the function $p(t)$ with the help of the optimal state.

We summarily analyse the problem of the synthesis of the control only in the case of the quadratic control. Suppose that the function $p(t)$ has the form

$$p(t) = -P(t)x(t) + r(t), \tag{10.5.1}$$

in which $P(t)$ is a matrix $n \times m-$dimensional and $r(t)$ is a scalar function. Both functions, and $P(t)$ and $r(t)$, are unknown and must be determinate.

We can reformulate the problem of the synthesis of the quadratic optimal control in this way:

Determine the matrix function $P(t)$ and the scalar function $r(t)$ such that the function $p(t)$ is expressed with the help of $P(t)$ and $r(t)$ by using the rule (10.5.1). We now remember the system of relations (from the maximum principle) satisfied by the optimal pair (x, u):

$$\begin{cases} x' = Ax + Bu + f \\ p' = -A^* p + Qx \\ B^* p \in 2Nu^* + \partial I_\Omega (u). \end{cases}$$

We substitute in this sistem the expression of p from (10.5.1) such that we are led to the following system:

$$\begin{cases} -p'x - Px' + r' = -A^*(-Px + r) + Qx \\ x' = Ax + B\left(-N^{-1}B^* Px + N^{-1}B^* r\right) + f. \end{cases} \tag{10.5.2}$$

From here we deduce:

$$\left(-P' - PA + PBN^{-1}B^*P - A^*P - Q\right)x + r' - A^*r - PBN^{-1}B^*r - Pf = 0.$$

We interpret the left-hand side term of this relation as a polynomial with the unknown variable x. Since the polynomial is null, we deduce that it has only null coefficients. So, we obtain the following two problems:

$$\begin{cases} p' + PA + A^*P - PDP = -Q \\ p(T) = 0 \end{cases} \tag{10.5.3}$$

and, respectivelly:

$$\begin{cases} r' = A^* r + PDr + Pf \\ r(T) = 0. \end{cases} \tag{10.5.4}$$

With the help of first problem we will determine the matrix function $P(t)$, and, with the help of the second problem we will determine the scalar function$r(t)$, after we have determinate the function $P(t)$. We have denoted by D the matrix $BN^{-1}B^*$.

Conversely, if we make the calculations in the opposite direction, we find that if the matrix function $P(t)$ satisfies the problem (10.5.3) and the scalar function $r(t)$ verifies the problem (10.5.4), then the functions $P(t)$ and $r(t)$ verify the relation (10.5.1) and this is equivalently to the fact that the pair $(x(t), u(t))$ is optimal.

Therefore, all reduces to show that the problem (10.5.3) has a solution, in the unknown $P(t)$, since if the function $P(t)$ is determinate, then the problem (10.5.4) has, without fail, a solution because the equation from the problem (10.5.4) is a linear equation. It is easy to see that the equation from the problem (10.5.4) is of the Ricatti type, and the problem is enframed with the general form of a Cauchy problem:

$$\begin{cases} x' = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

In certain conditions of regularity (for instance, in the conditions of the Picard's theorem), the problem (10.5.3) has a unique solution, and, therefore, we obtain the value of the function $P(t)$ in an interval of the form $[T - \varepsilon, T]$, which can be prolonged on the whole interval $[0, T]$.

After we determine the functions $P(t)$ and $r(t)$, the optimal control can be determined with the help of the formula:

$$u(t) = -N^{-1}B^*Px + N^{-1}B^*r,$$

which is called *the synthesis formula of the optimal control*.

Remarks 2. The matrix operator P and the scalar function r are depending only of the system, and do not depend on the chose of the initial conditions.

2. In the case of the quadratic control, the problem of the synthesis reduces to solve a Cauchy problem attached to a differential equation of the Ricatti type.

Application. Consider the problem of quadratic control:

$$\begin{cases} x' = x + u + 1 \\ x(0) = 1 \\ \min \int_0^1 \left(x^2 + u^2\right) \mathrm{d}t. \end{cases}$$

If we take the function $p(t)$ of the form $p(t) = -P(t)x(t) + r(t)$ then we obtain the following two problems:

$$\begin{cases} p' + 2p - p^2 + 1 = 0, \ t \in [0, 1) \\ p(1) = 0, \end{cases}$$

$$\begin{cases} r' = r + pr + p, \ t \in [0, 1) \\ p(1) = 0. \end{cases}$$

We solve the Ricatti's equation of the first problem:

$$\frac{\mathrm{d}p}{(p-1)^2 - 2} = \mathrm{d}t \Rightarrow \int \frac{\mathrm{d}p}{(p-1)^2 - 2} = t + C,$$

$$\frac{\mathrm{d}p}{(p-1)^2 - 2} = \frac{C_1}{p - 1 - \sqrt{2}} + \frac{C_2}{p - 1 - \sqrt{2}}.$$

We obtain:

$$\frac{1}{2\sqrt{2}} \ln\left(p - \sqrt{2} - 1\right) + \frac{1}{2\sqrt{2}} \ln\left(p + \sqrt{2} - 1\right) = t + C \Rightarrow$$

$$\Rightarrow \frac{p + \sqrt{2} - 1}{p - \sqrt{2} - 1} = Ce^{2\sqrt{2}t}.$$

Now, we add the initial condition $p(1) = 0$ and then we find $p(t)$.

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